A spectral sequence proof of Reisner's criterion

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Abstract

We prove Reisner's criterion using the two spectral sequences arising from the partition/Koszul double complex introduced in [Adi18].[1]

Let k be a field. Recall that an abstract simplicial complex Δ is said to be *Cohen-Macaulay* (over k) if its face ring $k^*[\Delta]$ is a Cohen-Macaulay ring. Reisner's criterion allows us to understand this important property on a geometric level:

Theorem (Reisner's criterion [Rei76])**.** *An abstract simplicial complex Δ is Cohen-Macaulay over k if and only if for every face $\sigma \in \Delta$, the link $\operatorname{Lk}_\sigma \Delta \subseteq \Delta$ satisfies*

$$\widetilde{H}^i(\operatorname{Lk}_\sigma \Delta; k) = 0, \qquad 0 \leq i < \dim \operatorname{Lk}_\sigma \Delta \tag{1}$$

Proof. Induction on $\dim \Delta = d - 1$. For $\dim \Delta = 0$, both the Cohen-Macaulay property and the vanishing condition (1) are always true.

Assume the statement is true in dimensions strictly less than $\dim \Delta > 0$. Then (1) holds if and only if[2]

(i) $\widetilde{H}^i(\Delta) = 0$ for $0 \leq i < \dim \Delta$ and

(ii) $\operatorname{St}_\sigma \Delta$ is Cohen-Macaulay for each $\varnothing \neq \sigma \in \Delta$.

[1] This approach to Reisner's theorem is due to Adiprasito.

[2] From now on, the field k is implicit: Simplicial homology is always taken with coefficients in k and Cohen-Macaulay means Cohen-Macaulay over k.

Indeed, for each $\sigma \neq \varnothing$ we have $\mathrm{St}_\sigma \Delta = \mathrm{Lk}_\sigma \Delta * \sigma$, so

$$k^*[\mathrm{St}_\sigma \Delta] \cong k^*[\mathrm{Lk}_\sigma \Delta] \otimes k^*[\sigma].$$

Hence $\mathrm{St}_\sigma \Delta$ is Cohen-Macaulay if and only if $\mathrm{Lk}_\sigma \Delta$ is Cohen-Macaulay. Since σ is nonempty, $\dim \mathrm{Lk}_\sigma \Delta = \dim \Delta - \dim \sigma - 1 < \dim \Delta$, so by the induction hypothesis $\mathrm{Lk}_\sigma \Delta$ is Cohen-Macaulay if and only if $\mathrm{Lk}_\tau (\mathrm{Lk}_\sigma \Delta) = \mathrm{Lk}_{\tau\sigma} \Delta$ has vanishing homology except in the top dimension for each $\tau \in \mathrm{Lk}_\sigma \Delta$, and as σ varies over the nonempty faces this condition is just (ii). Similarly, if Δ is assumed to be Cohen-Macaulay, then every star $\mathrm{St}_\sigma \Delta$ is again Cohen-Macaulay. Thus at the induction step we may assume that every proper star is Cohen-Macaulay.

We may also assume that Δ is of pure dimension. Indeed, Cohen-Macaulay local rings are of pure dimension, so Cohen-Macaulay complexes are pure-dimensional. If Δ satisfies (i) and (ii) then every proper star in Δ is Cohen-Macaulay and thus pure-dimensional. But $\widetilde{H}^0(\Delta) = 0$, so Δ must be of pure dimension.

It only remains to prove that for Δ of pure dimension and having all proper stars Cohen-Macaulay (i.e. for Δ a Buchsbaum complex), the Cohen-Macaulay property for Δ is equivalent to $\widetilde{H}^i(\Delta) = 0$ for $0 \leq i < \dim \Delta$.

Let $\mathcal{R}$ be the polynomial ring over the vertices of Δ and choose $\theta_1, \ldots, \theta_d \in \mathcal{R}^1 = k^1[\Delta]$ to be some linear system of parameters for Δ. Then $k^*[\Delta]$ is Cohen-Macaulay if and only if $H^i(K^\bullet \otimes_{\mathcal{R}} k^*[\Delta]) = 0$ for $i \neq d$, where $K^\bullet = K^\bullet(\theta_1, \ldots, \theta_d)$ is the Koszul (cochain) complex. Also, let $\widetilde{P}^\bullet$ denote the unreduced partition complex

$$0 \to k^*[\Delta] \to \bigoplus_{v \in \Delta^{(0)}} k^*[\mathrm{St}_v \Delta] \to \cdots \to \bigoplus_{F \in \Delta^{(d-1)}} k^*[\mathrm{St}_F \Delta] \to 0,$$

which equals the (reduced) simplicial cochain complex of Δ in monomial degree 0 (so we must put $k^*[\Delta]$ in cochain degree -1), but is exact in positive degrees. We have two spectral sequences for computing the homology of $\mathrm{Tot}^\bullet = \mathrm{Tot}^\bullet(K^\bullet \otimes_{\mathcal{R}} \widetilde{P}^\bullet)$, namely the column-wise spectral sequence

$${}'E_2^{s,t} = H_{II}(H_I^{s,t}(K^\bullet \otimes_{\mathcal{R}} \widetilde{P}^\bullet)) \Rightarrow H^{s+t}(\mathrm{Tot}^\bullet),$$

and the row-wise spectral sequence

$${}''E_2^{s,t} = H_I(H_{II}^{s,t}(K^\bullet \otimes_{\mathcal{R}} \widetilde{P}^\bullet)) \Rightarrow H^{s+t}(\mathrm{Tot}^\bullet).$$

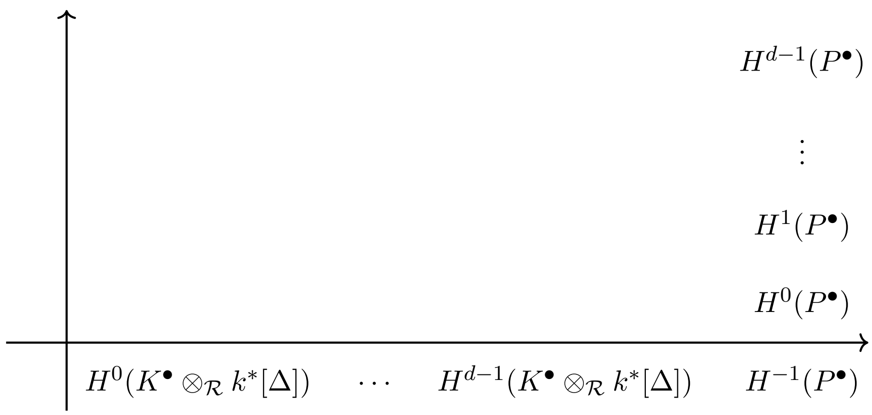
Since every proper star of Δ is Cohen-Macaulay, we have

$$H_I^{s,t}(K^\bullet \otimes_{\mathcal{R}} \widetilde{P}^\bullet) = \bigoplus_{f \in \Delta^{(t)}} H^s(K^\bullet \otimes_{\mathcal{R}} k^*[\operatorname{St}_f \Delta]) = 0$$

for $s \neq d$ and $t \geq 0$. For $s = d$, the complex $H_I^{d,\bullet}(K^\bullet \otimes_{\mathcal{R}} \widetilde{P}^\bullet)$ is the reduced partition complex

$$0 \to \mathscr{A}^*[\Delta] \to \bigoplus_{v \in \Delta^{(0)}} \mathscr{A}^*[\operatorname{St}_v \Delta] \to \cdots \to \bigoplus_{F \in \Delta^{(d-1)}} \mathscr{A}^*[\operatorname{St}_F \Delta] \to 0,$$

which we denote by $P^\bullet$. Hence ${}'E_2$ is:



so the only nonzero entries lie in the dth column or the -1st row. All differentials have trivial source or target, so the spectral sequence collapses on the second page. Note in particular that

$$H^i(\operatorname{Tot}^\bullet) \cong H^{i+1}(K^\bullet \otimes_{\mathcal{R}} k^*[\Delta]), \qquad 0 \leq i \leq d-2. \tag{2}$$

Turning now to the row-wise spectral sequence, note that $K^s \otimes_{\mathcal{R}} \widetilde{P}^t \to K^{s+1} \otimes_{\mathcal{R}} \widetilde{P}^t$ maps vertical cocycles to vertical cocycles of positive monomial degree, which must be vertical coboundaries since $\widetilde{P}^\bullet$ is exact in positive monomial degrees. Hence the differentials in $H_{II}^{\bullet,\bullet}(K^\bullet \otimes_{\mathcal{R}} \widetilde{P}^\bullet)$ are zero, so

$${}''E_2^{s,t} = H_I(H_{II}^{s,t}(K^\bullet \otimes_{\mathcal{R}} \widetilde{P}^\bullet)) = H_{II}^{s,t}(K^\bullet \otimes_{\mathcal{R}} \widetilde{P}^\bullet) \cong K^s \otimes_{\mathcal{R}} \widetilde{H}^t(\Delta) \cong \widetilde{H}^t(\Delta)^{\binom{d}{d-s}}.$$

By a similar diagram chase, the differentials on ${}''E_2$ are all zero, so ${}''E_\bullet$ collapses on the second page. Since we are over a field there are no extension problems, so $''E_\bullet$ gives

$$H^i(\mathrm{Tot}^\bullet) \cong \bigoplus_{j=0}^{i} \widetilde{H}^j(\Delta)^{\binom{d}{d-i+j}}. \tag{3}$$

Comparing (2) and (3), we find

$$H^{i+1}(K^\bullet \otimes_{\mathcal{R}} k^*[\Delta]) \cong \bigoplus_{j=0}^{i} \widetilde{H}^j(\Delta)^{\binom{d}{d-i+j}}, \tag{4}$$

from which the conclusion follows immediately. □

Remark. In [Adi18], the spectral sequences associated with the double complex $K^\bullet \otimes_{\mathcal{R}} \widetilde{P}^\bullet$ (with $k = \mathbb{R}$) are used to calculate

$$H^{-1}(P^\bullet) = \ker \left(\mathscr{A}^*(\Delta) \to \bigoplus_{v \in \Delta^{(0)}} \mathscr{A}^*(\mathrm{St}_v \Delta) \right)$$

for Δ a Buchsbaum complex by observing from $'E_2$ and (3) that

$$H^{-1}(P^\bullet) \cong H^{d-1}(\mathrm{Tot}^\bullet) \cong \bigoplus_{j=0}^{d-1} \widetilde{H}^j(\Delta)^{\binom{d}{j+1}}.$$

Here $H^{-1}(P^\bullet)$ comes with a grading, and by tracking monomial degrees one finds that the $\widetilde{H}^j(\Delta)^{\binom{d}{j+1}}$-summand corresponds to the $(j+1)$st graded piece, so

$$\ker \left(\mathscr{A}^j(\Delta) \to \bigoplus_{v \in \Delta^{(0)}} \mathscr{A}^j(\mathrm{St}_v \Delta) \right) \cong H^{j-1}(\Delta)^{\binom{d}{j}}.$$

Remark. Keeping track of monomial degrees, the isomorphism (4) implies the following extension of Reisner's theorem: If Δ is an $(n-2)$-acyclic Buchsbaum complex (for some $n \leq d$), then any linear system of parameters $\theta_1, \ldots, \theta_d \in k^1[\Delta]$ is regular up to degree n.

References

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