WHITEHEAD IMPLIES CHOICE

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The proof of Whitehead's theorem (and related statements) is usually referred to as a "cell induction", but implicitly uses the axiom of choice. The typical proof uses the axiom of choice in two ways: (i) for the n-th step in the skeletal "induction", a nulhomotopy is chosen for every n-cell and (ii) when the proof "builds a function inductively", this implicitly uses the axiom of dependent choice. The point of this note is to show that the axiom of choice is unavoidable.

For X a possibly infinite set, we let $F(X)^\bullet$ denote the semisimplicial set whose n-simplices are ordered subsets of X of length n, endowed with the obvious face maps. Then $F(X)^\bullet$ is the so-called "complex of injective words", studied e.g. in [Far78] and showing up more recently in [RW11].

Theorem 1. *(i) If X is finite, then the geometric realization $\|F(X)^\bullet\|$ is homotopy equivalent to a wedge of $(|X|-1)$-spheres.*
(ii) If X is infinite, then $\|F(X)^\bullet\|$ is weakly contractible.

Proof. (i) is Proposition 3.2 in [RW11]. (ii) follows from (i) since for compactness reasons, any map $S^n \to \|F(X)^\bullet\|$ factors through a subcomplex $\|F(X_0)^\bullet\| \subseteq \|F(X)^\bullet\|$, where $X_0 \subseteq X$ is a finite subset having $|X_0| \geq n+2$. □

Theorem 2. *Whitehead's theorem implies the axiom of choice.*

Proof. Let $(X_i)_{i\in I}$ be a family of non-empty sets indexed by some set I. For each i, let $Y_i = \mathbb{N} \times X_i$. Thus $\|F(Y_i)^\bullet\|$ is weakly contractible by Theorem 1. There is then an obvious weak homotopy equivalence $\coprod_{i\in I} \|F(Y_i)^\bullet\| \to I$ which collapses $\|F(Y_i)^\bullet\|$ to i. (Here I is

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topologized as a discrete space.) Assuming Whitehead's theorem, this map has a homotopy inverse $f\colon I \to \coprod_{i\in I} \|F(Y_i)^\bullet\|$. For each i, we then have $f(i) \in \|F(Y_i)^\bullet\|$, so $f(i)$ lies in the relative interior of a unique cell of $\|F(Y_i)^\bullet\|$, which by construction corresponds to some ordered set $y_{k_1} < \cdots < y_{k_r}$ of elements in Y_i. Projecting $y_{k_1} \in Y_i = \mathbb{N} \times X_i$ onto X_i, we get an element $x_i \in X_i$. This defines a choice function. □

References

[Far78] Frank D Farmer, *Cellular homology for posets*, Math. Japon. **23** (1978), no. 6, 79.

[RW11] Oscar Randal-Williams, *Homological stability for unordered configuration spaces*, Quart. J. Math. **64** (2011), no. 1, 303–326.