Six-functor formalism for topological spaces

Oscar Harr

January 3, 2021

Contents

Overview

These are notes for my TopTop presentation on Verdier's six-functor formalism for locally compact Hausdorff spaces. The original reference is [Ver65]. Akhil Mathew's well-written notes [Mat11] provide a helpful companion to this paper.

In the spirit of TopTop, these notes will take a modern approach to the six-functor formalism for locally compact Hausdorff spaces. In [Lur14], Lurie explains how his covariant Verdier duality theorem [Lur17, 5.5.5.4] can be used to produce the two most difficult functors $f_!$ and $f^!$ from a map of locally compact Hausdorff spaces f.[1] We will cover this in Sections 1 and 2, but leave the proof of covariant Verdier duality to Section 6. The remaining sections are spent fleshing out as much of the full six-functor formalism as I've been able to. The results are summarized in Theorem 5.1.

Comments and corrections are welcome!

A note on ambiguity. The term "canonical" is ambiguous in mathematics, but is useful for other things besides giving the reader (or perhaps the author) a false sense of security. In these notes, it always indicates that the object referred to as such (1) has been or (2) will be uniquely defined up to contractible ambiguity. Here are two representative examples:

(1) In the definition of $\mathcal{C}$-valued sheaves given below, we consider the "canonical map $\mathcal{F}(U) \to \varprojlim_V \mathcal{F}(V)$". Implicitly, this refers to any map belonging to the contractible space of maps coming from the cone $\{\mathcal{F}(U) \to \mathcal{F}(V)\}_V$ via the universal property of limits.

(2) In Proposition 2.2, I write: "There is a canonical natural transformation $f_! \to f_*$". This means that I will give a recipe for the intended object which only depends on a contractible space of choices (often the recipe is in the proof). When I later refer to "the canonical natural transformation $f_! \to f_*$", this ensures that the object appearing in your head and the object appearing in my head will belong to the same contractible space of choices.

1 (co)Sheaves

Let $\mathcal{C}$ be an ∞-category admitting all small limits and colimits.

Definition. For a topological space X, a $\mathcal{C}$*-valued sheaf* on X is a presheaf $\mathcal{F} \in \mathrm{PSh}_{\mathcal{C}}(X) = \mathrm{Fun}(\mathcal{U}(X)^{\mathrm{op}}, \mathcal{C})$ such that for each cover $\{U_\alpha \to U\}$, the canonical map

$$\mathcal{F}(U) \to \varprojlim_V \mathcal{F}(V)$$

[1] Here $f_!$ corresponds to the functor $Rf_!$ from the old approach to Verdier duality. The old $f_!$ is constructed explicitly by letting $f_!\mathcal{F}$ be the subsheaf of $f_*\mathcal{F}$ consisting of sections with proper support over the base. This $f_!$ is generally neither a left adjoint nor a right adjoint, but it is by construction *left exact*, so it makes sense to consider the right derived functor $Rf_!$. Verdier then proves that $Rf_!$ has a right adjoint $f^!$, which is the same as our $f^!$.

is an equivalence, where V ranges over open subsets contained in some U_α. Let $\mathrm{Sh}_{\mathcal{C}}(X) \subseteq \mathrm{PSh}_{\mathcal{C}}(X)$ be the full subcategory spanned by sheaves.

Remark. This agrees with the general definition of sheaves on an ∞-category equipped with a Grothendieck topology (HTT 6.2.2) because $\mathcal{U}(X)$ is small, so every covering sieve is stupidly generated by a cover (e.g. the cover consisting of all elements of the sieve).

Proposition 1.1. The inclusion $\mathrm{Sh}_{\mathcal{C}}(X) \subseteq \mathrm{PSh}_{\mathcal{C}}(X)$ has a left adjoint, called *sheafification*. If filtered colimits in $\mathcal{C}$ are left exact (e.g. if $\mathcal{C}$ is stable), then sheafification is left exact.

Proof. Lurie constructs the sheafification of space-valued presheaves in the proof of [Lur04, 9.0.6], but the construction only uses the existence of small limits and (filtered) colimits, whereas exactness precisely uses that filtered colimits are exact. □

Corollary 1.2. If $\mathcal{C}$ is stable, then $\mathrm{Sh}_{\mathcal{C}}(X)$ is stable.

Proof. Filtered colimits in $\mathcal{C}$ are left exact, so the proposition implies that $\mathrm{Sh}_{\mathcal{C}}(X)$ is a left exact localization of the stable ∞-category $\mathrm{PSh}_{\mathcal{C}}(X)$. □

If $\mathcal{C}$ is presentable, we have an improved version of Proposition 1.1 which will be used later. Recall that a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ of a presentable ∞-category is *strongly reflective* if $\mathcal{C}_0$ is stable under equivalences in $\mathcal{C}$ and the inclusion $i\colon \mathcal{C}_0 \subseteq \mathcal{C}$ admits a left adjoint $L\colon \mathcal{C} \to \mathcal{C}_0$ such that the endofunctor $iL\colon \mathcal{C} \to \mathcal{C}$ is accessible. Equivalently, $\mathcal{C}_0 \subseteq \mathcal{C}$ is a reflective subcategory which is stable under equivalences and such that $\mathcal{C}_0$ is again presentable.

Proposition 1.3. If $\mathcal{C}$ is presentable, then $\mathrm{Sh}_{\mathcal{C}}(X)$ is a strongly reflective subcategory of $\mathrm{PSh}_{\mathcal{C}}(X)$. In particular, $\mathrm{Sh}_{\mathcal{C}}(X)$ is presentable.

Proof. This follows from [Lur09b, 5.5.4.(17-19)]. □

Remark. It is tempting to assume that $\mathcal{C}$ is presentable from the get-go and replace Proposition 1.1 by Proposition 1.3. But below we consider sheaves valued in $\mathcal{C}^{\mathrm{op}}$, and it is very rarely the case that both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are presentable ∞-categories, so a theory of sheaves valued in presentable ∞-categories would be insufficient for our purposes, even though we are eventually only interested in such $\mathcal{C}$. When $\mathcal{C}$ is stable (as it mostly will be in these notes), the situation is especially bad. Suppose that $\mathcal{C}$ is stable with both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ presentable. Then the homotopy categories $h\mathcal{C}$ and $h(\mathcal{C}^{\mathrm{op}}) \simeq (h\mathcal{C})^{\mathrm{op}}$ are again presentable. But since $\mathcal{C} \to h\mathcal{C}$ preserves products and coproducts, the 1-category $h\mathcal{C}$ also has a zero object, so by [GU71, 7.13] $h\mathcal{C}$ is equivalent to the terminal category. Hence all objects of $\mathcal{C}$ are equivalent, and since $\mathcal{C}$ has a zero object we have in fact that all objects of $\mathcal{C}$ are zero objects, so $\mathcal{C}$ is also equivalent to the terminal category.

A continuous map $f\colon Y \to X$ induces a functor $f^{-1}\colon \mathcal{U}(X)^{\mathrm{op}} \to \mathcal{U}(Y)^{\mathrm{op}}$, which in turn gives a precomposition functor $(f^{-1})^*\colon \mathrm{PSh}_{\mathcal{C}}(Y) \to \mathrm{PSh}_{\mathcal{C}}(X)$. This functor has left and right adjoints given by (pointwise!) left and right Kan extension respectively. Following the notation in [Lur09b, 4.3.3], we write these adjunctions as

$$\mathrm{PSh}_{\mathcal{C}}(Y) \underset{(f^{-1})_*}{\overset{(f^{-1})_!}{\underset{\longrightarrow}{\overset{(f^{-1})^*}{\longleftarrow\!\!\longrightarrow}}}} \mathrm{PSh}_{\mathcal{C}}(X)$$

The functor $(f^{-1})^*$ maps sheaves to sheaves. We let $f_*\colon \mathrm{Sh}_{\mathcal{C}}(Y) \to \mathrm{Sh}_{\mathcal{C}}(X)$ denote its restriction to sheaves. We refer to this functor as *pushforward*. It fits into an adjunction

$$\mathrm{Sh}_{\mathcal{C}}(Y) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Sh}_{\mathcal{C}}(X)$$

where f^* is the so-called *pullback* functor defined by

$$\mathrm{Sh}_{\mathcal{C}}(X) \subseteq \mathrm{PSh}_{\mathcal{C}}(X) \xrightarrow{(f^{-1})_!} \mathrm{PSh}_{\mathcal{C}}(Y) \to \mathrm{Sh}_{\mathcal{C}}(Y),$$

where the last functor is sheafification.

Remark. From now on, the symbol f_* (resp. f^*) can either mean right Kan extension along f (resp. precomposition by f) or the pushforward (resp. pullback) map that we just defined depending on whether f is a morphism in the category of topological spaces or a morphism in the category of simplicial sets (which is how we consider f^{-1} above). Later, I will also use the lower shriek symbol $f_!$ to denote the exceptional pushforward functor (to be defined) if f is a morphism of topological spaces. If the reader type checks morphisms, these notational overlaps should not cause any confusion. (The star-shriek notation for Kan extensions is justified by the six-functor formalism for simplicial sets – and it looks better than clunky symbols involving Lan and Ran.)

We will also need the following simple fact:

Proposition 1.4. Let $j\colon V \hookrightarrow X$ be an open embedding. Then j^* is given by restriction and j_* is fully faithful.

Proof. Clearly restriction satisfies the pointwise definition of Kan extensions [Lur09b, 4.3.3.2] since for $U \subseteq V$ open, the poset $\{W \subseteq U,\ U \text{ open in } X\}$ has U as its terminal object.

With respect to this restriction model of j^*, the counit $j^*j_* \to 1$ is given pointwise by identities. Hence it is a natural equivalence. This implies that j_* is fully faithful. □

Definition. The ∞-category of $\mathcal{C}$-*valued cosheaves* on $\mathcal{C}$ is $\mathrm{coSh}_{\mathcal{C}}(X) = \mathrm{Sh}_{\mathcal{C}^{\mathrm{op}}}(X)^{\mathrm{op}}$.

Pushforward and pullback of $\mathcal{C}^{\mathrm{op}}$-valued sheaves induces an adjunction

$$\mathrm{coSh}_{\mathcal{C}}(Y) \underset{f^+}{\overset{f_+}{\rightleftarrows}} \mathrm{coSh}_{\mathcal{C}}(X)$$

by taking opposites (reversing the roles of left and right adjoint!).

2 Covariant Verdier duality

Let $\mathcal{C}$ be a stable ∞-category admitting all small limits and colimits. In [Lur17, 5.5.5], Lurie shows:

Theorem 2.1 (Verdier duality)**.** Let X be a locally compact Hausdorff space. Then there is a canonical adjoint equivalence

$$\mathrm{Sh}_{\mathcal{C}}(X) \simeq \mathrm{coSh}_{\mathcal{C}}(X),$$

which we will denote by D_X or, if multiple coefficient categories are being considered, by $D_{X,\mathcal{C}}$.

Concretely, the equivalence is given as follows. Every presheaf $\mathcal{F} \in \mathrm{PSh}_{\mathcal{C}}(X)$ has an associated co-presheaf of compactly supported sections $\mathcal{F}_c \in \mathrm{PSh}_{\mathcal{C}^{\mathrm{op}}}(X)$ given by

$$\mathcal{F}_c \colon U \mapsto \varinjlim_{K \subseteq U} \left(\mathrm{fib}\left(\mathcal{F}(X) \to \mathcal{F}(X - K)\right)\right),$$

where K ranges over compact subsets of U. Hausdorff-ness ensures that $X - K$ is open, so $\mathcal{F}(X - K)$ makes sense. This (contravariant!) construction defines a functor $\mathrm{PSh}_{\mathcal{C}}(X) \to \mathrm{PSh}_{\mathcal{C}^{\mathrm{op}}}(X)^{\mathrm{op}}$. The equivalence $D_{X,\mathcal{C}}$ is the restriction of this construction to sheaves. (In particular, $\mathcal{F} \mapsto \mathcal{F}_c$ sends sheaves to cosheaves.)

Given a map $f\colon Y \to X$ of locally compact Hausdorff spaces, Verdier duality now produces an adjunction $D_X^{-1} f_+ D_Y =: f_! \dashv f^! := D_Y^{-1} f^+ D_X$, where D_X^{-1} and D_Y^{-1} are picked from the contractible spaces of inverses to D_X and D_Y. Thus f gives rise to adjunctions

$$\mathrm{Sh}_{\mathcal{C}}(Y) \underset{f^!}{\overset{f^*}{\leftrightarrows}} \mathrm{Sh}_{\mathcal{C}}(X) \quad (f^* \dashv f_*,\ f_! \dashv f^!)$$

such that all four adjoints can be viewed as values of f under functors (contravariant in the case of f^* and $f^!$) from LCHaus to the ∞-category of stable ∞-categories. In particular, for composable maps $Z \xrightarrow{g} Y \xrightarrow{f} X$, there are canonical equivalences

$$(fg)_! \simeq f_! g_! \quad \text{and} \quad (fg)^! \simeq g^! f^!$$

Remark. In the classical situation $\mathcal{C} = D(A)$, A a commutative ring, these facts were historically shown using the Leray spectral sequence. In Lurie's approach, which we have taken here, they follow directly from definitions.

A common feature of six-functor formalisms is that the so-called *exceptional* functors $f_!$ and $f^!$, which may be difficult to understand in general, will sometimes coincide with their ordinary counterparts.

Proposition 2.2. There is a canonical natural transformation $\mathrm{Nm}_f \colon f_! \to f_*$ which is an equivalence if f is proper.

Proof. Equivalently, there is a natural transformation $f_+D_Y \to D_X f_*$ defined uniquely up to contractible choice, such that $f_+D_Y \to D_X f_*$ is an equivalence when f is proper.

For $Z \subseteq Y$ closed and $\mathcal{F} \in \mathrm{Sh}_{\mathcal{C}}(Y)$, let $\mathcal{F}_Z$ denote the fiber $\mathrm{fib}(\mathcal{F}(Y) \to \mathcal{F}(Y-Z))$. Then by definition $(f_+D_Y\mathcal{F})(V) = \varinjlim_K \mathcal{F}_K$ and $(D_X f_*\mathcal{F})(V) = \varinjlim_L \mathcal{F}_{f^{-1}L}$, with K ranging over compact subsets of $f^{-1}V$ and L ranging over compact subsets of V. The "corestriction" maps $\mathcal{F}_K \to \mathcal{F}_{f^{-1}(fK)}$, which come from restricting $\mathcal{F}(Y-K) \to \mathcal{F}(Y - f^{-1}(fK))$, induce a map $(f_+D_Y\mathcal{F})(V) \to (D_X f_*\mathcal{F})(V)$ which is compatible with corestrictions and natural in $\mathcal{F}$, thus giving the desired natural transformation $f_+D_Y \to D_X f_*$. If f is proper, then each $f^{-1}(fK)$ is again a compact subset of $f^{-1}V$ and the maps $\mathcal{F}_K \to \mathcal{F}_{f^{-1}(fK)}$ are transition maps, hence induce an equivalence upon taking colimits. □

A result in the same style holds for upper shriek:

Proposition 2.3. Let $j \colon V \hookrightarrow X$ be an open embedding. Then there is a canonical natural equivalence $j^! \simeq j^*$. Furthermore, $j_!$ is given by

$$(j_!\mathcal{F})(U) = \begin{cases} \mathcal{F}(U) & \text{if } U \subseteq V, \\ 0, & \text{otherwise,} \end{cases}$$

with restrictions and functoriality given in the obvious way.

Proof. For the first statement, it suffices to construct a natural equivalence $j^+D_X \simeq D_V j^*$ depending only on contractible choices. Let $\mathcal{F} \in \mathrm{Sh}_{\mathcal{C}}(X)$, $U \subseteq V$ open. Then for each $K \subseteq U$ compact, note that since $X - K \supseteq U - K \subseteq U$ is cofinal in the poset $\{W \subseteq X \text{ open} \mid W \subseteq X - K \text{ or } W \subseteq U\}$ (hence final upon taking opposites), the sheaf condition for the cover $\{X-K, U\}$ of X implies that

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(X-K) & \longrightarrow & \mathcal{F}(U-K) \end{array}$$

is a pullback. Hence the induced map on fibers

$$\mathrm{fib}\left(\mathcal{F}(X) \to \mathcal{F}(X-K)\right) \to \mathrm{fib}\left(\mathcal{F}(V) \to \mathcal{F}(V-K)\right)$$

(uniquely defined up to contractible choice) is an equivalence. Taking colimits over K on both sides, we get an equivalence

$$(j^+D_X\mathcal{F})(U) \to (D_V j^*\mathcal{F})(U)$$

(again uniquely defined up to contractible choice) which is compatible with corestriction maps and natural in $\mathcal{F}$.

The fact that $j^! = j^*$ is given by restriction can also be written $j^* = (\underline{j})^*$, where $\underline{j} \colon \mathcal{U}(V)^{\mathrm{op}} \to \mathcal{U}(X)^{\mathrm{op}}$ is the map of posets defined by $\underline{j}(U) = jU = U$. But then the left adjoint $j_!$ is left Kan extension along $\underline{j}$ which is given by the formula in the second statement. □

3 Base change

Classically, base change is shown using universality of effaceable cohomological δ-functors (see [Mat11]). This approach is not available in the general setup of the previous sections. Nonetheless, Lurie has shown:

Theorem 3.1 (Nonabelian proper base change)**.** Let $\mathcal{C}$ be an ∞-category admitting all small limits and colimits and having enough compact objects. Given a Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{g} & X \end{array} \tag{3.1}$$

of locally compact Hausdorff spaces with p proper, the canonical natural transformation

$$g^*p_* \to p'_*p'^*g^*p_* \simeq p'_*g'^*p^*p_*, \to p'_*g'^*$$

is a natural equivalence of functors $\mathrm{Sh}_{\mathcal{C}}(Y) \to \mathrm{Sh}_{\mathcal{C}}(X')$.

Proof. This is [Lur09b, 7.3.1.19]. □

Remark. Here Cartesian means Cartesian in the (1-)category of topological spaces. Since locally compact Hausdorff spaces are compactly generated, there is no room for confusion even if you are used to k-ifying every topological space in sight.

Remark. In the literature, the map $g^*p_* \to p'_*g'^*$ is called the *Beck–Chevalley morphism* (or simply the *base-change morphism*) associated to the square (3.1). Note that for us it is uniquely defined only up to contractible choice.

In this section, we will recover the familiar general base change result involving lower shriek functors by piggybacking on Lurie's theorem. The only other input is an easy lemma:

Lemma 3.2. Let $f\colon Y \to X$ be a continuous map and let $j\colon V \hookrightarrow X$ be an open embedding. Form the pullback

$$\begin{array}{ccc} f^{-1}(V) & \xrightarrow{f_V} & V \\ \big\downarrow{\scriptstyle j'} & & \big\downarrow{\scriptstyle j} \\ Y & \xrightarrow{f} & X \end{array}$$

Then the canonical natural transformation

$$j^*f_* \to j^*f_*j'_*j'^* \simeq j^*j_*(f_V)_*j'^* \to (f_V)_*j'^*$$

is an equivalence.

Proof. Using that j^* and j'^* are given by restricting, the claim can be checked directly. □

Remark. Using Lemma 2.3, we can also write this equivalence as $j^! f_* \xrightarrow{\sim} (f_V)_* j'^!$.

Theorem 3.3 (Base change)**.** Given a Cartesian diagram of locally compact Hausdorff spaces

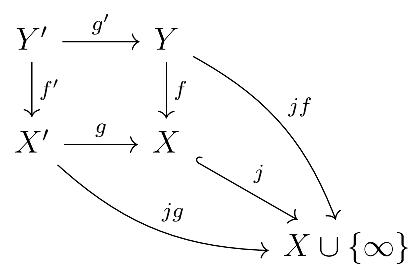
$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

then there are canonical equivalences

$$g^* f_! \xrightarrow{\sim} f'_! g'^* \qquad \text{and} \qquad f^! g_* \xrightarrow{\sim} g'_* f'^!,$$

Proof. Giving one equivalence (uniquely up to contractible choice) is equivalent to giving the other (uniquely up to contractible choice).

We first reduce to the case of X compact. Let $j\colon X \hookrightarrow X \cup \{\infty\}$ be the one-point compactification of X. This is an open embedding since X is locally compact Hausdorff. In particular, it is injective, so below the outer square is again Cartesian:



Assume base change holds for the outer diagram. Then we have canonical equivalences

$$g'_* f'^! \xleftarrow{\sim} (jf)^! (jg)_* = f^! j^* j_* g_* \xrightarrow{\sim} f^! g_*, \tag{3.2}$$

where we have used that $j^! = j^*$ by Lemma 2.3 and the last map comes from contracting $j^* j_* \to 1$, which is an equivalence since the right adjoint j_* is fully faithful. But (3.2) is precisely base change for the inner square. Thus we may assume that X is compact.

If X is compact, the map $f\colon Y \to X$ factors uniquely through the Stone-Čech compactification $Y \hookrightarrow \beta Y$, which is an open embedding since Y is locally compact Hausdorff. Thus we have a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'_1} & & \downarrow{\scriptstyle j} \\ Z & \xrightarrow{h} & \beta Y \\ \downarrow{\scriptstyle f'_2} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{g} & X \end{array}$$

where both inner squares and the outer square (which is the square we eventully care about) are Cartesian. But here p is trivially proper since it is a map of Hausdorff spaces with compact domain. Thus base change holds for the lower square by Theorem 3.1 and for the upper square by Lemma 3.2. Base change for the outer square now follows by an easy diagram chase. □

4 Monoidal structure

In this section, we produce the remaining two functors of the six-functor formalism by showing that if $\mathcal{C}^\otimes$ is a presentably symmetric monoidal ∞-category, then $\mathrm{Sh}_\mathcal{C}(X)$ has a canonical (presentably) symmetric monoidal structure.

As in the classical setting, the symmetric monoidal structure on sheaves arises by sheafifying the pointwise symmetric monoidal structure on presheaves. Here we will need two basic results which I cannot find in the holy texts.

First, an obvious symmetric analogue of a remark that can be found "somewhere in Lurie" [Lur09a, 1.1.18]:

Proposition 4.1. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category. Then for any simplicial set K there is a symmetric monoidal structure on $\mathrm{Fun}(K, \mathcal{C})$, called the *pointwise symmetric monoidal structure*, whose underlying tensor product is equivalent to

$$\mathrm{Fun}(K,\mathcal{C}) \times \mathrm{Fun}(K,\mathcal{C}) \cong \mathrm{Fun}(K, \mathcal{C}\times\mathcal{C}) \xrightarrow{(-\otimes-)_*} \mathrm{Fun}(K,\mathcal{C}),$$

where $- \otimes -$ is any choice of tensor product on $\mathcal{C}$.

Proof. Let $\mathcal{C}^\otimes \to \mathrm{Fin}_*$ be the coCartesian fibration encoding the symmetric monoidal structure on $\mathcal{C}$. Postcomposition gives a coCartesian fibration $\mathrm{Fun}(K, \mathcal{C}^\otimes) \to \mathrm{Fun}(K, \mathrm{Fin}_*)$. Form the pullback

$$\begin{array}{ccc} \mathrm{Fun}(K,\mathcal{C})^\otimes & \longrightarrow & \mathrm{Fun}(K,\mathcal{C}^\otimes) \\ \downarrow & & \downarrow \\ \mathrm{Fin}_* & \longrightarrow & \mathrm{Fun}(K,\mathrm{Fin}_*) \end{array}$$

where the bottom map sends $\langle n\rangle$ to the constant functor at $\langle n\rangle$. Then the lefthand map is again a coCartesian fibration, Note that by construction $\mathrm{Fun}(K,\mathcal{C})^\otimes_{\langle n\rangle} = \mathrm{Fun}(K, \mathcal{C}^\otimes_{\langle n\rangle})$, and in particular $\mathrm{Fun}(K,\mathcal{C})^\otimes_{\langle 1\rangle} = \mathrm{Fun}(K,\mathcal{C})$.

A morphism in $\mathrm{Fun}(K,\mathcal{C})^\otimes$ is coCartesian if and only if it is pointwise coCartesian [Lur09b, 3.1.2.1], so a straightening $\mathrm{Fin}_* \to \widehat{\mathbf{Cat}}_\infty$ of $\mathrm{Fun}(K,\mathcal{C}^\otimes) \to \mathrm{Fin}_*$ maps $f\colon \langle m\rangle \to \langle n\rangle$ to postcomposition by the image of f under some straightening of $\mathcal{C}^\otimes \to \mathrm{Fin}_*$. Hence the underlying tensor product of the pointwise monoidal structure is as desired. □

In particular, the proposition promotes $\mathrm{PSh}_\mathcal{C}(X)$ to a symmetric monoidal ∞-category $\mathrm{PSh}_\mathcal{C}(X)^\otimes$ for any topological space X.

Proposition 4.2. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category and let $f\colon K \to L$ be a map of simplicial sets. Then

(1) Precomposition $f^*\colon \operatorname{Fun}(L,\mathcal{C}) \to \operatorname{Fun}(K,\mathcal{C})$ can be given the structure of a symmetric monoidal functor with respect to the pointwise monoidal structures.

(2) If in addition tensoring $-\otimes -\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ preserves colimits in each variable, then left Kan extension $f_!\colon \operatorname{Fun}(K,\mathcal{C}) \to \operatorname{Fun}(L,\mathcal{C})$ can also be promoted to a symmetric monoidal functor (if it exists).

Proof. Using the construction from the proof of Proposition 4.1, note that precomposition $f^*\colon \operatorname{Fun}(L,\mathcal{C}^\otimes) \to \operatorname{Fun}(K,\mathcal{C}^\otimes)$ restricts to a functor $\operatorname{Fun}(L,\mathcal{C})^\otimes \to \operatorname{Fun}(L,\mathcal{C})^\otimes$ mapping coCartesian edges to coCartesian edges since coCartesian-ness is checked pointwise. The restriction to fibers over $\langle 1\rangle$ is exactly $f^*\colon \operatorname{Fun}(L,\mathcal{C}) \to \operatorname{Fun}(K,\mathcal{C})$. This proves (1).

As for (2), recall that the left Kan extension $f_!$ is left adjoint to f^* [Lur09b, 4.3.3.7]. By standard nonsense (e.g. [Lur17, 7.3.2]), we can then equip $f_!$ with the structure of a lax symmetric monoidal functor, i.e. we can promote it to a morphism $f_!^\otimes\colon \operatorname{Fun}(K,\mathcal{C})^\otimes \to \operatorname{Fun}(L,\mathcal{C})^\otimes$ of ∞-operads. The cocontinuity assumption on $-\otimes-$ now implies that $f_!^\otimes$ has the *property* of being symmetric monoidal (since left Kan extensions are computed pointwise by colimits). □

For the remainder of this section, we assume that $\mathcal{C}^\otimes$ is a *presentably symmetric monoidal ∞-category*, meaning that $\mathcal{C}$ is presentable and tensoring preserves colimits in each variable. Equivalently, $\mathcal{C}^\otimes$ is an $\mathbb{E}_\infty$-algebra in $\mathbf{Pr}^{L,\otimes}$, where the symmetric monoidal structure on $\mathbf{Pr}^{L,\otimes}$ is the Lurie tensor product. In particular, these assumptions guarantee that for every $M\in\mathcal{C}$ (and some choice of tensor product), the endofunctor $-\otimes M$ has an (essentially unique) right adjoint $\underline{\operatorname{Hom}}(M,-)$.

Proposition 4.3. The pointwise symmetric monoidal structure makes $\operatorname{PSh}_{\mathcal{C}}(X)^\otimes$ presentably symmetric monoidal for each topological space X.

Proof. Presentability is [Lur09b, 5.5.3.6] and cocontinuity of the pointwise tensor product comes from cocontinuity of the tensor product in $\mathcal{C}$ and the fact that colimits in $\operatorname{PSh}_{\mathcal{C}}(X)$ are computed pointwise. □

Thus $\operatorname{PSh}_{\mathcal{C}}(X)^\otimes$ has its own internal Hom, which we denote by $\mathcal{H}om$. We will use the following fact:

Lemma 4.4. If $\mathcal{F}\in\operatorname{PSh}_{\mathcal{C}}(X)$ and $\mathcal{F}'\in\operatorname{Sh}_{\mathcal{C}}(X)$, then $\mathcal{H}om(\mathcal{F},\mathcal{F}')\in\operatorname{Sh}_{\mathcal{C}}(X)$.

Proof. For each open $U\subseteq X$, let $R_U\colon \operatorname{PSh}_{\mathcal{C}}(X)\to\operatorname{PSh}_{\mathcal{C}}(X)$ denote the obvious endofunctor with

$$(R_U\mathcal{G})(V) = \mathcal{G}(U\cap V).$$

Note that an inclusion $V\subseteq U$ induces a restriction map $R_U\mathcal{G}\to R_V\mathcal{G}$. Since limits in $\operatorname{PSh}_{\mathcal{C}}(X)$ are computed pointwise, the sheaf condition on $\mathcal{G}$ is equivalent to requiring that the map

$$R_U\mathcal{G} \to \varprojlim_V R_V\mathcal{G} \tag{4.1}$$

is an equivalence for any cover $\{U_\alpha \to U\}$ and with V ranging over the covering sieve generated by this cover.

The endofunctor R_U has a left adjoint L_U with

$$(L_U\mathcal{G})(V) = \begin{cases} \mathcal{G}(V), & V \subseteq U, \\ \varnothing_{\mathcal{C}}, & \text{otherwise.} \end{cases}$$

(Here $\varnothing_{\mathcal{C}}$ denotes an initial object in $\mathcal{C}$.) Since the tensor product is assumed to be cocontinuous in each variable, we have in particular $\varnothing_{\mathcal{C}} \otimes c \simeq \varnothing_{\mathcal{C}}$ for each $c \in \mathcal{C}$. But then for arbitrary presheaves $\mathcal{G}, \mathcal{G}'$, we have canonical equivalences

$$L_U\mathcal{G} \otimes \mathcal{G}' \simeq L_U(\mathcal{G} \otimes \mathcal{G}') \simeq \mathcal{G} \otimes L_U\mathcal{G}'. \tag{4.2}$$

We now use (4.1) to check that $\mathcal{H}om(\mathcal{F}, \mathcal{F}')$ is a sheaf. For every presheaf $\mathcal{G}$, the map $R_U\mathcal{H}om(\mathcal{F}, \mathcal{F}') \to \varprojlim_V R_V\mathcal{H}om(\mathcal{F}, \mathcal{F}')$ induces

$$\begin{aligned} \mathrm{Map}(\mathcal{G}, \varprojlim\nolimits_V R_V\mathcal{H}om(\mathcal{F}, \mathcal{F}')) &\simeq \varprojlim\nolimits_V \mathrm{Map}(L_U\mathcal{G}, \mathcal{H}om(\mathcal{F}, \mathcal{F}')) \\ &\simeq \varprojlim\nolimits_V \mathrm{Map}(L_U(\mathcal{G} \otimes \mathcal{F}), \mathcal{F}') \\ &\simeq \mathrm{Map}(\mathcal{G} \otimes \mathcal{F}, \varprojlim\nolimits_V R_V\mathcal{F}') \\ &\simeq \mathrm{Map}(\mathcal{G} \otimes \mathcal{F}, R_U\mathcal{F}') \\ &\simeq \mathrm{Map}(\mathcal{G} \otimes L_U\mathcal{F}, \mathcal{F}') \\ &\simeq \mathrm{Map}(\mathcal{G}, R_U\mathcal{H}om(\mathcal{F}, \mathcal{F}')), \end{aligned}$$

and the conclusion follows by the Yoneda lemma. □

Let $p\colon \mathrm{PSh}_{\mathcal{C}}(X)^{\otimes} \to \mathrm{Fin}_*$ denote the coCartesian fibration encoding the pointwise symmetric monoidal structure on $\mathrm{PSh}_{\mathcal{C}}(X)$. Define $\mathrm{Sh}_{\mathcal{C}}(X)^{\otimes}$ to be the full subcategory of $\mathrm{PSh}_{\mathcal{C}}(X)^{\otimes}$ spanned by objects of the form $\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_n$ for $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathrm{Sh}_{\mathcal{C}}(X)$, in the notation of [Lur17, 2.1.1.15].

Proposition 4.5. The restriction $p \,|\, \mathrm{Sh}_{\mathcal{C}}(X)^{\otimes}\colon \mathrm{Sh}_{\mathcal{C}}(X)^{\otimes} \to \mathrm{Fin}_*$ is a coCartesian fibration, making $\mathrm{Sh}_{\mathcal{C}}(X)$ a presentably symmetric monoidal ∞-category. The adjunction

$$\mathrm{Sh}_{\mathcal{C}}(X) \underset{i}{\overset{L}{\rightleftarrows}} \mathrm{PSh}_{\mathcal{C}}(X)$$

where L is sheafification, can be promoted to an adjunction

$$\mathrm{Sh}_{\mathcal{C}}(X)^{\otimes} \underset{i^{\otimes}}{\overset{L^{\otimes}}{\rightleftarrows}} \mathrm{PSh}_{\mathcal{C}}(X)^{\otimes}$$

where $L^{\otimes}$ is monoidal and $i^{\otimes}$ is lax monoidal.

Proof. Denote the sheafification functor by L. Using the theory of monoidal localizations [Lur17, 2.2.1], it suffices to show that if $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ is an L-equivalence of presheaves and $\mathcal{F}$ is any presheaf, then $\varphi \otimes \mathcal{F}\colon \mathcal{F}_1 \otimes \mathcal{F} \to \mathcal{F}_2 \otimes \mathcal{F}$ is again an L-equivalence. To see this, let $\mathcal{G} \in \mathrm{Sh}_{\mathcal{C}}(X)$ be any sheaf. Adjunction gives a commutative diagram of mapping spaces

$$\begin{array}{ccc} \mathrm{Map}(\mathcal{F}_2 \otimes \mathcal{F}, \mathcal{G}) & \xrightarrow{(\varphi\otimes\mathcal{F})^*} & \mathrm{Map}(\mathcal{F}_1 \otimes \mathcal{F}, \mathcal{G}) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Map}(\mathcal{F}_2, \mathcal{H}om(\mathcal{F}, \mathcal{G})) & \xrightarrow{\varphi^*} & \mathrm{Map}(\mathcal{F}_1, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \end{array}$$

Here the bottom arrow is an equivalence by Lemma 4.4 and the assumption that φ is an L-equivalence. Hence the top arrow is an equivalence by two out of three. Since $\mathcal{G}$ was an arbitrary sheaf, it follows that $\varphi \otimes \mathcal{F}$ is an L-equivalence as desired. □

Proposition 4.6. Let $f\colon Y \to X$ be a map of spaces. Then the adjunction

$$\mathrm{Sh}_{\mathcal{C}}(Y) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathrm{Sh}_{\mathcal{C}}(X)$$

can be given the structure of a monoidal adjunction, promoting f^* to a symmetric monoidal functor and f_* to a lax monoidal functor.

Proof. Denote the sheafification functors by L_Y and L_X and the right adjoint inclusions by i_Y and i_X, so that by definition $f^* = L_Y(f^{-1})_! i_X$. Propositions 4.2 and 4.2 promote f^* to a map of ∞-operads $(f^*)^{\otimes} = L_Y^{\otimes}(f^{-1})_!^{\otimes} i_X^{\otimes}$. It remains to show that $(f^*)^{\otimes}$ carries coCartesian edges to coCartesian edges.

Note that an edge φ of $\mathrm{Sh}_{\mathcal{C}}(X)^{\otimes}$ is coCartesian if and only if it is equivalent to an edge of the form $L_X^{\otimes}\psi$ where ψ is a coCartesian edge of $\mathrm{PSh}_{\mathcal{C}}(Y)^{\otimes}$. A quick calculation shows that the canonical transformation $L_Y(f^{-1})_! i_X L_X \to L_Y(f^{-1})_!$ is an equivalence, and hence that $L_Y^{\otimes}(f^{-1})_!^{\otimes} i_X^{\otimes} L_X^{\otimes} \to L_Y^{\otimes}(f^{-1})_!^{\otimes}$ is an equivalence. But then

$$(f^*)^{\otimes} L_X^{\otimes}\psi = L_Y^{\otimes}(f^{-1})_!^{\otimes} i_X^{\otimes} L_X^{\otimes}\psi \simeq L_Y^{\otimes}(f^{-1})_!^{\otimes}\psi$$

is coCartesian since $L_Y^{\otimes}(f^{-1})_!^{\otimes}$ is symmetric monoidal. □

5 Summary and applications

Let LCHaus denote the category of locally compact Hausdorff spaces. Collecting the results of Sections 2, 3 and 4 (and with a little extra work), we get (cf. [GR17, p. 273]):

Theorem 5.1. Let $\mathcal{C}^{\otimes}$ be a stable presentably symmetric monoidal ∞-category whose underlying category $\mathcal{C}$ has enough compact objects. Then there are functors $\mathrm{LCHaus}^{\mathrm{op}} \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathbf{Pr}^{\mathrm{L},\otimes})$ and $\mathrm{LCHaus} \to \mathbf{Pr}^{\mathrm{L}}$ given on objects by $X \mapsto \mathrm{Sh}_{\mathcal{C}}(X)$ and on morphisms by $f \mapsto f^*$ (right adjoint: f_*) and $f \mapsto f_!$ (right adjoint: $f^!$) respectively.

For any map $f\colon Y \to X$ in LCHaus, we have

(1) (Enriched adjunction) There are equivalences

$$f_*\mathcal{H}om(f^*\mathcal{G}, \mathcal{F}) \simeq \mathcal{H}om(\mathcal{G}, f_*\mathcal{F}) \tag{5.1}$$

natural in $\mathcal{F} \in \mathrm{Sh}_{\mathcal{C}}(Y)$, $\mathcal{G} \in \mathrm{Sh}_{\mathcal{C}}(X)$.

(2) (Proper adjunction) There is a canonical natural transformation

$$f_! \to f_*$$

which is an equivalence if f is proper.

(3) (Open adjunction) Let $j\colon V \hookrightarrow Y$ be an open embedding. There is a canonical natural equivalence $j^! \simeq j^*$.

(4) (Base change) Given a Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

there are canonical equivalences.

$$g^* f_! \xrightarrow{\sim} f'_! g'^* \qquad \text{and} \qquad f^! g_* \xrightarrow{\sim} g'_* f'^!$$

Assume further that $\mathcal{C}^{\otimes}$ is rigid in the sense of [GR17, p. 79]. Then we also have

(1) (Enriched adjunctions$^+$) There are equivalences

$$f_*\mathcal{H}om(\mathcal{F}, f^!\mathcal{G}) \simeq \mathcal{H}om(f_!\mathcal{F}, \mathcal{G}) \tag{5.2}$$

natural in $\mathcal{F} \in \mathrm{Sh}_{\mathcal{C}}(Y)$, $\mathcal{G} \in \mathrm{Sh}_{\mathcal{C}}(X)$.

(5) (Projection formula) Suppose X has finite covering dimension. The functor $f_!\colon \mathrm{Sh}_{\mathcal{C}}(Y) \to \mathrm{Sh}_{\mathcal{C}}(X)$ can be given the structure of a map of $\mathrm{Sh}_{\mathcal{C}}(X)^{\otimes}$-modules, where the module structure on $\mathrm{Sh}_{\mathcal{C}}(Y)$ comes from the monoidal map f^*. In particular, this means that there are equivalences

$$f_!(\mathcal{F} \otimes f^*\mathcal{G}) \simeq f_!\mathcal{F} \otimes \mathcal{G} \tag{5.3}$$

natural in $\mathcal{F} \in \mathrm{Sh}_{\mathcal{C}}(Y)$, $\mathcal{G} \in \mathrm{Sh}_{\mathcal{C}}(X)$.

Proof. Items (2), (3) and (4) are exactly Propositions 2.2, 2.3, and 3.3.

The equivalence (5.1) follows formally from symmetric monoidality of f^*. Indeed, for any sheaf $\mathcal{L} \in \mathrm{Sh}_{\mathcal{C}}(Y)$, we have equivalences

$$\begin{aligned} \mathrm{Map}(\mathcal{L}, \mathcal{H}om(\mathcal{F}, f_*\mathcal{G})) &\simeq \mathrm{Map}(\mathcal{L} \otimes \mathcal{F}, f_*\mathcal{G}) \\ &\simeq \mathrm{Map}(f^*\mathcal{L} \otimes f^*\mathcal{F}, \mathcal{G}) \\ &\simeq \mathrm{Map}(\mathcal{L}, f_*\mathcal{H}om(f^*\mathcal{F}, \mathcal{G})) \end{aligned}$$

natural in $\mathcal{L}$, $\mathcal{F}$ and $\mathcal{G}$. Similarly, (5.2) will follow once we know (5.3) via the natural equivalences

$$\begin{aligned} \mathrm{Map}(\mathcal{L}, \mathcal{H}om(f_!\mathcal{F}, \mathcal{G})) &\simeq \mathrm{Map}(\mathcal{L} \otimes f_!\mathcal{F}, \mathcal{G}) \\ &\simeq \mathrm{Map}(f_!(f^*\mathcal{L} \otimes \mathcal{F}), \mathcal{G}) \\ &\simeq \mathrm{Map}(f^*\mathcal{L} \otimes \mathcal{F}, f^!\mathcal{G}) \\ &\simeq \mathrm{Map}(\mathcal{L}, f_*\mathcal{H}om(\mathcal{F}, f^!\mathcal{G})). \end{aligned}$$

As for (5), first note that if $i\colon X \hookrightarrow X \cup \{\infty\}$, then $f_! = i^*(if)_!$ since i is an open embedding, so using the description of $i_!$ in Proposition 2.3 one sees that $i^*i_! \simeq 1$. Since i^* is a map of modules, it will suffice to show that $(if)_!$ is a map of modules, i.e. we have reduced to the case of X compact. If X is compact, the universal property of Stone-Čech compactification allows us to factor f as $Y \xrightarrow{j} \beta Y \xrightarrow{p} X$, where j is an open embedding and p is proper. It suffices to see that $j_!$ and $p_!$ are strict functors of $\mathrm{Sh}_{\mathcal{C}}(X)^{\otimes}$-modules.

By (3), $j_!$ is left adjoint to $j^! = j^*$ which comes with a canonical structure of a map of $\mathrm{Sh}_{\mathcal{C}}(\beta Y)^{\otimes}$-modules. General nonsense implies that $j_!$ gets a structure of a left-lax map of $\mathrm{Sh}_{\mathcal{C}}(\beta Y)^{\otimes}$-modules (e.g. [GR17, p. 41]). Strictness of this left-lax structure follows from the explicit description of $j_!$ given in Proposition 2.3. Being a functor of $\mathrm{Sh}_{\mathcal{C}}(\beta Y)^{\otimes}$-modules implies that $j_!$ is in particular a functor of $\mathrm{Sh}_{\mathcal{C}}(X)^{\otimes}$-modules.

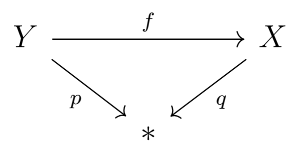
Similarly, $p_! = p_*$ gets a structure of a right-lax functor of $\mathrm{Sh}_{\mathcal{C}}(X)^{\otimes}$-modules. Hence it suffices to see that the natural transformation $p_*\mathcal{F} \otimes \mathcal{G} \to p_*(\mathcal{F} \otimes p^*\mathcal{G})$ coming from adjunction is an equivalence. Theorem 7.2.3.6 and Corollary 7.2.1.12 in [Lur09b] imply that $\mathrm{Sh}_{\mathcal{C}}(X)$ is hypercomplete, so equivalences of sheaves are detected stalkwise. Using this fact and proper base change (which by construction is compatible with the right-lax structures on proper pushforwards), we are reduced to the case $X = \{*\}$. But here $\mathrm{Sh}_{\mathcal{C}}(X)^{\otimes} \simeq \mathcal{C}^{\otimes}$ is rigid, so by [GR17, p. 83] the right-lax structure on p_* is automatically strict. □

Example. Let $\mathcal{C}^{\otimes} = \mathrm{LMod}_A^{\otimes}$ be the symmetric monoidal ∞-categories of A-modules for some $A \in \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sptr})$. Then $\mathcal{C}^{\otimes}$ is a rigid [GR17, p. 79] presentably symmetric monoidal stable ∞-category. Thus the full version of Theorem 5.1 holds for these coefficient categories. By [Lur17, 7.1.1.16], this includes the classical examples $\mathcal{C}^{\otimes} = D(A)^{\otimes}$ for A a discrete commutative ring.

Verdier's six-functor formalism – or rather, the cheaper four functor formalism of Sections 2 and 3 – gives us a handy way to understand homology (plain and Borel–Moore) and cohomology (plain and compactly supported) of locally compact Hausdorff spaces. Namely, let $M \in \mathcal{C}$ where $\mathcal{C}$ is a stable ∞-category admitting all small limits and colimits. If $p\colon X \to *$ is the projection from a locally compact Hausdorff space to a point, then we have the following dictionary:

cohomology with coefficients in M	p_*p^*M
homology with ——″——	$p_!p^!M$
compactly supported cohomology with ——″——	$p_!p^*M$
Borel–Moore homology with ——″——	$p_*p^!M$

This dictionary offers a convenient way to understand the functoriality of all four theories. A map $f\colon Y \to X$ of locally compact Hausdorff spaces gives a commutative diagram



Using the unit $1 \to f_*f^*$ and counit $f_!f^! \to 1$, we get canonical natural transformations

$$p_*p^* = (qf)_*(qf)^* \simeq q_*f_*f^*q^* \leftarrow q_*q^* \quad \text{and} \quad p_!p^! = (qf)_!(qf)^! \simeq q_!f_!f^!q^! \to q_!q^!$$

capturing respectively the contravariance of cohomology and covariance of homology. If f is proper, then the same unit and counit give

$$p_!p^* = (qf)_!(qf)^* \simeq q_!f_*f^*q^* \leftarrow q_!q^* \quad \text{and} \quad p_*p^! = (qf)_*(qf)^! \simeq q_*f_!f^!q^! \to q_*q^!,$$

showing that compactly supported cohomology is contravariant in proper maps and Borel-Moore homology is covariant in proper maps.

In the same way that Grothendieck duality generalizes Serre duality, the adjunction (5.2) is a generalization of Poincaré duality to locally compact Hausdorff spaces. Spelling this out, note that if $\mathbf{1}$ denotes the unit of $\mathcal{C}^{\otimes}$, then for any locally compact Hausdorff space $p\colon X \to *$, we have

$$\underline{\mathrm{Hom}}(p_!p^*\mathbf{1}, \mathbf{1}) \simeq \underline{\mathrm{Hom}}(\mathbf{1}, p_*p^!\mathbf{1}) \simeq p_*p^!\mathbf{1}, \tag{5.4}$$

i.e. Borel-Moore homology is the dual of compactly supported cohomology. If X is a d-dimensional manifold and $\mathcal{C}^{\otimes} = D(\mathbb{Z})^{\otimes}$, a local computation shows that $p^!\mathbf{1} \simeq \omega_X[d]$ where ω_X is the classical *orientation sheaf* of X. (See [Mat11, 5.5] for details.) By definition X is orientable if and only if $\omega_X \simeq p^*\mathbf{1}$, where choosing an equivalence $\omega_X \simeq p^*\mathbf{1}$ is the same as choosing an orientation of X. Hence the calculation (5.4) shows that $\underline{\mathrm{Hom}}(p_!p^*\mathbf{1}, \mathbf{1}) \simeq p_*p^*\mathbf{1}[d]$. Finally, X is compact if and only if p is proper, in which case $p_* \simeq p_!$, so we arrive at the most familiar version of Poincare duality: if X is a compact orientable d-manifold, then $\underline{\mathrm{Hom}}(p_*p^*\mathbf{1}, \mathbf{1}) \simeq p_*p^*\mathbf{1}[d]$.

6 Proof of covariant Verdier duality

In this section we prove Theorem 2.1 following Lurie's proof in [Lur17, 5.5.5].

Let $\mathcal{C}$ be a stable ∞-category admitting all small limits and colimits, and let X be a locally compact Hausdorff space. A key ingredient in the proof of Verdier duality is that we can model $\mathcal{C}$-valued sheaves on X by functors defined on compact subsets of X rather than open subsets. For this model, the sheaf condition becomes:

Definition. Let $\mathcal{K}(X)$ denote the poset of compact subsets of X. A $\mathcal{K}$-sheaf on X is a functor $\mathcal{F} \in \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}}, \mathcal{C})$ satisfying:

(1) $\mathcal{F}(\varnothing) = 0$.

(2) For $K_1, K_2 \in \mathcal{K}(X)$, the diagram

$$\begin{array}{ccc} \mathcal{F}(K_1 \cup K_2) & \longrightarrow & \mathcal{F}(K_1) \\ \downarrow & & \downarrow \\ \mathcal{F}(K_2) & \longrightarrow & \mathcal{F}(K_1 \cap K_2) \end{array}$$

is a pullback square.

(3) For each $K \in \mathcal{K}(X)$, the canonical map

$$\varinjlim_L \mathcal{F}(L) \to \mathcal{F}(K)$$

is an equivalence, where L ranges over compact subsets of X such that $K \subseteq U \subseteq L$ for some open U in X.

Let $\mathrm{Sh}_{\mathcal{C}}^{\mathcal{K}}(X)$ denote the full subcategory of $\mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}}, \mathcal{C})$ spanned by $\mathcal{K}$-sheaves.

Theorem 6.1. For $\mathcal{F} \in \mathrm{Fun}((\mathcal{U}(X) \cup \mathcal{K}(X))^{\mathrm{op}}, \mathcal{C})$, the following conditions are equivalent:

(1) The restriction $\mathcal{F}_{\mathrm{cpt}} = \mathcal{F} \,|\, \mathcal{K}(X)^{\mathrm{op}}$ is a $\mathcal{K}$-sheaf and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}_{\mathrm{cpt}}$.

(2) The restriction $\mathcal{F}_{\mathrm{open}} = \mathcal{F} \,|\, \mathcal{U}(X)^{\mathrm{op}}$ is a sheaf and $\mathcal{F}$ is a left Kan extension of $\mathcal{F}_{\mathrm{open}}$.

Furthermore, let $\mathcal{A} \subseteq \mathrm{Fun}((\mathcal{U}(X) \cup \mathcal{K}(X))^{\mathrm{op}}, \mathcal{C})$ denote the full subcategory spanned by functors satisfying the equivalent conditions (1) and (2). Then the restriction maps

$$\mathrm{Sh}_{\mathcal{C}}(X) \leftarrow \mathcal{A} \to \mathrm{Sh}_{\mathcal{C}}^{\mathcal{K}}(X)$$

are trivial Kan fibrations.

Proof. The equivalence of conditions (1) and (2) is [Lur09b, 7.3.4.9]. The last statement now follows via [Lur09b, 4.3.2.15]. □

A similar approach is used in the proof of Verdier duality, i.e. the equivalence D_X will arise as a zigzag of restriction maps which have adjoint inverses given by Kan extension functors.

Let M be the poset of pairs (S,i), $0 \le i \le 2$ and $S \subseteq X$ with S compact if $i=0$, $X-S$ compact if $i=2$, and with order given by $(S,i) \le (T,j)$ if $S \subseteq T$ and $i \le j$ or if $i=0$ and $j=2$. With respect to the projection onto the second factor $M \to [2]$, note that $M_0 = \mathcal{K}(X)$ and $M_1 \cong \mathcal{K}(X)^{\mathrm{op}}$ via $(S,2) \mapsto X-S$.

Lemma 6.2. For $\mathcal{F} \in \mathrm{Fun}(M, \mathcal{C})$, the following conditions are equivalent:

(1) The restriction $(\mathcal{F}\,|\,M_0)^{\mathrm{op}} \colon \mathcal{K}(X)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a $\mathcal{K}$-sheaf, $\mathcal{F}\,|\,M_1 = 0$, and $\mathcal{F}$ is a left Kan extension of $\mathcal{F}\,|\,M_0 \cup M_1$.

(2) The restriction $\mathcal{F}\,|\,M_2 \colon \mathcal{K}(X)^{\mathrm{op}} \to \mathcal{C}$ is a $\mathcal{K}$-sheaf, $\mathcal{F}\,|\,M_1 = 0$, and $\mathcal{F}$ is a right Kan extension of $\mathcal{F}\,|\,M_1 \cup M_2$.

Modulo the lemma, we can then prove Verdier duality:

Proof of Theorem 2.1. Let $\mathcal{E} \subseteq \mathrm{Fun}(M, \mathcal{C})$ denote the full subcategory spanned by functors satisfying the equivalent conditions of the lemma. Using [Lur09b, 4.3.2.15] and Theorem 6.1, we have a zigzag of restriction maps

$$\mathrm{Sh}_{\mathcal{C}}(X) \leftarrow \mathcal{A} \to \mathrm{Sh}^{\mathcal{K}}_{\mathcal{C}}(X) \leftarrow \mathcal{E} \to \mathrm{coSh}^{\mathcal{K}}_{\mathcal{C}}(X) \leftarrow \mathcal{A}' \to \mathrm{coSh}_{\mathcal{C}}(X), \tag{6.1}$$

all of which are equivalences with adjoint inverses given by Kan extending in the appropriate direction. Here $\mathcal{A}$ is as in the statement of Theorem 6.1 with $\mathcal{A}'$ is defined analogously and $\mathrm{coSh}^{\mathcal{K}}_{\mathcal{C}}(X) = \mathrm{Sh}^{\mathcal{K}}_{\mathcal{C}^{\mathrm{op}}}(X)$. The formula $\mathcal{F} \mapsto \mathcal{F}_c$ for the resulting equivalence is found by chasing through (6.1) using the pointwise formulas for Kan extensions. □

Sketch of proof of Lemma 6.2. It suffices to show (2) implies (1) since the other direction is symmetric via the order-reversing self-bijection $M \to M$ given by $(S,i) \mapsto (X-S, 2-i)$. For this direction, it is convenient to enlarge M by also allowing $(S,2)$ with $X-S$ open. Denote the larger poset by M'. With respect to the projection $M' \to [2]$, we now have $M'_i = M_i$ for $i = 0,1$ and $M'_2 = (\mathcal{U}(X) \cup \mathcal{K}(X))^{\mathrm{op}}$. The point of working with this larger poset is that we can outsource some hard work to Theorem 6.1. Namely, let $\mathcal{B} \subseteq \mathrm{Fun}(M', \mathcal{C})$ be the full subcategory spanned by functors satisfying

(i) $\mathcal{F}\,|\,M_2$ is a $\mathcal{K}$-sheaf,

(ii) $\mathcal{F}\,|\,M'_2$ is a right Kan extension of $\mathcal{F}\,|\,M_2$,

(iii) $\mathcal{F}\,|\,M_1 = 0$, and

(iv) $\mathcal{F}$ is a right Kan extension of $\mathcal{F}\,|\,M_1 \cup M_2$.

Applying [Lur09b, 4.3.2.15], we have that the restriction $\mathcal{B} \to \mathcal{E}$ is a trivial Kan fibration, so in particular it has a section (given by Kan extending in this case). Hence it will suffice to show that for any $\mathcal{F} \in \mathcal{B}$, $\mathcal{F} \,|\, M$ satisfies condition (1). But from Theorem 6.1 and using [Lur09b, 4.3.2.15] again, restriction also gives a trivial Kan fibration $\mathcal{B} \to \mathrm{Sh}_{\mathcal{C}}(X)$. For $\mathcal{F} \in \mathcal{B}$ restricting to $\mathcal{G} \in \mathrm{Sh}_{\mathcal{C}}(X)$, one finds by chasing around using the pointwise formulas for Kan extensions that $\mathcal{F}_0 = \mathcal{F} \,|\, M_0$ is given by $\mathcal{F}_0(K) = \mathrm{fib}\,(\mathcal{G}(X) \to \mathcal{G}(X - K))$. To verify condition (1), we first check that $\mathcal{F}_0^{\mathrm{op}} \colon \mathcal{K}(X)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a $\mathcal{K}$-sheaf. There are three things to check:

(i) $\mathcal{F}_0(\varnothing) = \mathrm{fib}(\mathcal{G}(X) \to \mathcal{G}(X - \varnothing)) = 0$.

(ii) For $K_1, K_2 \subseteq X$ compact, we must show that

$$\begin{array}{ccc} \mathcal{F}_0(K_1 \cap K_2) & \longrightarrow & \mathcal{F}_0(K_1) \\ \downarrow & & \downarrow \\ \mathcal{F}_0(K_2) & \longrightarrow & \mathcal{F}_0(K_1 \cup K_2) \end{array} \tag{6.2}$$

is a pullback square in $\mathcal{C}^{\mathrm{op}}$ or equivalently a pushout square in $\mathcal{C}$. But we can write this square as the fiber of

$$\begin{tikzcd} \mathcal{G}(X) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathcal{G}(X) \arrow[dd] \arrow[dr] & \\ & \mathcal{G}(X - (K_1 \cap K_2)) \arrow[rr] \arrow[dd] & & \mathcal{G}(X - K_1) \arrow[dd] \\ \mathcal{G}(X) \arrow[rr] \arrow[dr] & & \mathcal{G}(X) \arrow[dr] & \\ & \mathcal{G}(X - K_2) \arrow[rr] & & \mathcal{G}(X - (K_1 \cup K_2)) \end{tikzcd}$$

which is a map of pullback squares, so (6.2) is again a pullback square and hence a pushout square since $\mathcal{C}$ is stable.

(iii) For the third condition, note that since limits preserve fiber sequences, we have for each $K \in \mathcal{K}(X)$ the following map of fiber sequences:

$$\begin{array}{ccc} \mathcal{F}_0(K) & \longrightarrow & \varprojlim_{K \Subset L} \mathcal{F}_0(L) \\ \downarrow & & \downarrow \\ \mathcal{G}(X) & \longrightarrow & \varprojlim_{K \Subset L} \mathcal{G}(X) \\ \downarrow & & \downarrow \\ \mathcal{G}(X - K) & \longrightarrow & \varprojlim_{K \Subset L} \mathcal{G}(X - L) \end{array}$$

where $K \Subset L$ is shorthand for the condition that there is an open U in X with $K \subseteq U \subseteq L$. Since $\{X - L \mid K \Subset L\}$ is a covering sieve of $X - K$, the bottom map is an equivalence. The middle horizontal map is an equivalence since the poset $\{L \mid K \Subset L\}$ is filtered. Hence the uppermost map is an equivalence as desired.

Hence $\mathcal{F}_0$ is a $\mathcal{K}$-sheaf. It remains to be seen that $\mathcal{F} \mid M$ is a left Kan extension of $\mathcal{F} \mid M_0 \cup M_1$. By [Lur09b, 4.3.2.8], it suffices to show that $\mathcal{F} \mid M$ is a left Kan extension of $\mathcal{F} \mid M''$, where $M'' \subseteq M$ is the subposet defined by only allowing $(S, 1)$ with S compact. Furthermore, let $B \subseteq M_2'$ be the subposet of pairs $(S, 2)$ having $X - S$ open with compact closure. Again using [Lur09b, 4.3.2.8], it will be enough to show

(i) $\mathcal{F} \mid M'' \cup M_2'$ is a left Kan extension of $\mathcal{F} \mid M'' \cup B$,

(ii) $\mathcal{F} \mid M'' \cup B$ is a left Kan extension of $\mathcal{F} \mid M''$.

For (i), note that $\mathcal{F} \mid M_2'$ is a left Kan extension of $\mathcal{F} \mid M'''$, where $M''' \subseteq M_2'$ is the subposet consisting of $(S, 2)$ with $X - S$ compact or open with compact closure. This follows from Theorem 6.1 and the fact that for compact K, the open $U \supseteq K$ with $\overline{U}$ compact are cofinal in the poset of neighborhoods of K. Thus it suffices to observe that the inclusion $M'''_{/(X-K,2)} \subseteq M'' \cup M'''_{/(X-K,2)}$ is cofinal. This can be checked using Quillen's Theorem A [Lur09b, 4.1.3.1].

It remains to prove (ii). For this, let $(S, 2) \in B$. We wish to show that $\mathcal{F}(S, 2)$ is a colimit of $\mathcal{F} \mid M''_{/(S,2)}$. For $K \subseteq X$ compact, let M''_K be the subposet of M'' formed by (L, i) with $(K, 0) \leq (L, i) \leq (S, 2)$. Note that $M''_{/(S,2)}$ is a filtered colimit of the simplicial sets M''_K for $K \supseteq X - S$, so by general nonsense we can compute $\varinjlim \mathcal{F} \mid M''_{/(S,2)}$ as $\varinjlim \{\varinjlim_K \mathcal{F} \mid M''_K\}_{K \supseteq X-K}$. Let $U = X - S$. But the span $(K, 0) \leftarrow (K - U, 0) \rightarrow (K - U, 1)$ is cofinal in M''_K, so we need only show that the corresponding diagram

$$\begin{array}{ccc}
\mathcal{F}(K-U,0) & \longrightarrow & \mathcal{F}(K-U,1) \\
\downarrow & & \downarrow \\
\mathcal{F}(K,0) & \longrightarrow & \mathcal{F}(S,2)
\end{array}$$

is a pushout. Consider the larger diagram

$$\begin{array}{ccccc}
\mathcal{F}(K-U,0) & \longrightarrow & \cancel{\mathcal{F}(K-U,1)} & & \\
\downarrow & & \downarrow & & \\
\mathcal{F}(K,0) & \longrightarrow & T & \longrightarrow & \cancel{\mathcal{F}(K,1)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}(\varnothing,2) & \longrightarrow & \mathcal{F}(K-U,2) & \longrightarrow & \mathcal{F}(K,2) \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{F}(X-U,2) & \longrightarrow & \cancel{\mathcal{F}(X,2)}
\end{array}$$

where T is the pullback of the middle right square. Here we have used that $\mathcal{F}\,|\,M_1 = 0$ and $\mathcal{F}(X,2) = \mathcal{G}(\varnothing) = 0$ to cancel out three corners. By the description of $\mathcal{F}_0 = \mathcal{F}\,|\,M_0$ given at the start of the proof, we see that the middle horizontal rectangle and left verticle rectangles are pullbacks (i.e. they are fiber sequences). Hence by general nonsense the left middle square and uppermost square are pullback. It now suffices to see that the composition $T \to \mathcal{F}(X-U,2) \to \mathcal{F}(X-U,2)$ is an equivalence. For this it suffices to see that the right verticle rectangle is a pullback square, and since we have already seen that its upper square is a pullback, we need only show that the lowermost square is a pullback. But this square is exactly

$$\begin{array}{ccc} \mathcal{G}(X-K\cup U) & \longrightarrow & \mathcal{G}(X-K) \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \longrightarrow & \mathcal{G}(\varnothing) \end{array}$$

which is a pullback since $\mathcal{G}$ is a sheaf. □

References

[GR17] Dennis Gaitsgory and Nick Rozenblyum, *A Study in Derived Algebraic Geometry*, vol. 1, American Mathematical Soc., 2017.

[GU71] Peter Gabriel and Friedrich Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Mathematics, vol. 221, Springer-Verlag Berlin Heidelberg, 1971.

[Lur04] Jacob Lurie, *Derived Algebraic Geometry*, Ph.D. thesis, Massachusetts Institute of Technology, 2004.

[Lur09a] ______, *Derived Algebraic Geometry II: Noncommutative Algebra*, URL: `https://math.mit.edu/~lurie/papers/DAG-II.pdf`, 2009.

[Lur09b] ______, *Higher Topos Theory*, Annals of Mathematics Studies, Princeton University Press, 2009.

[Lur14] ______, *Verdier duality (Lecture 21)*, URL:`http://people.math.harvard.edu/~lurie/282ynotes/LectureXXI-Verdier.pdf`, 2014.

[Lur17] ______, *Higher Algebra*, URL: `https://www.math.ias.edu/~lurie/`, 2017.

[Mat11] Akhil Mathew, *Verdier duality*, URL: `http://math.uchicago.edu/~amathew/verd.pdf`, 2011.

[Ver65] Jean-Louis Verdier, *Dualité dans la cohomologie des espaces localement compacts*, Séminaire Bourbaki **9** (1965), 337–349.