TWISTED HOMOLOGY STABILITY OF O_n FOR VALUATION RINGS

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Abstract. In this article, we extend an argument of Vogtmann in order to show homology stability of $O_n(A)$ when A is a valuation ring subject to arithmetic conditions on either its residue or its quotient field. In particular, it is shown that if A is a henselian valuation ring, then the groups $O_n(A)$ exhibit homology stability if the residue field of A has finite Pythagoras number. Our results include those of Vogtmann, and hold with various twisted coefficients. Using these results, we give analogues for fields $F \neq \mathbb{R}$ of some computations that appear in the study of scissor congruences.

0. Introduction

Let R be a commutative ring. For each $n \in \mathbb{Z}_{\geq 1}$, consider the quadratic modules

(Euclidean n-space) $\qquad \mathbb{E}^n_R = (R^n, n\langle 1\rangle),$

(hyperbolic $2n$-space) $\qquad \mathbb{H}^{2n}_R = (R^{2n}, n\langle 1, -1\rangle).$

(We leave out subscripts when no ambiguity may arise.) Here $n\langle 1\rangle$ denotes the Euclidean quadratic form $\sum_{i=1}^n X_i^2$ and $n\langle 1,-1\rangle$ is the hyperbolic quadratic form $\sum_{i=1}^{2n}(-1)^{i+1}X_i^2$. We write

(orthogonal group) $\qquad O_n(R) = O(n\langle 1\rangle),$

(split-orthogonal group) $\qquad O_{n,n}(R) = O(n\langle 1,-1\rangle)$

for the isometry groups of these modules.* The homology of split-orthogonal groups has been intensely studied due to its importance in hermitian K-theory (recalled in Section 3.6), and it is known to stabilize for a large class of rings [Vog81, Bet87, Cha87, MvdK01].

Less is known about the homology of Euclidean orthogonal groups, which has only been shown to stabilize over fields with certain arithmetic properties [Vog82]; over the rings of S-integers in number fields for certain sets of places S [Col11]; and for finite rings in which two is invertible [WK22]. Some work has been done on improving Vogtmann's stability range for $O_n(F)$ when F is a field subjected to stricter arithmetic conditions, e.g. by Cathelineau [Cat07] for infinite Pythagorean fields and Sprehn–Wahl [SW20] for fields whose Stufe is less than or equal to two.

In this article, we extend Vogtmann's methods to valuation rings. Our proof fits into the stability framework of Randal-Williams–Wahl [RWW17], and thus we also get homology stability with various twisted coefficients. Before we state our theorem, we recall some algebraic notions:

- The *Pythagoras number* $P(R)$ of a ring R is the smallest $p \in \mathbb{Z}_{\geq 1}$ so that every sum of squares in R is a sum of p squares, if any such number p exists; if no such number exists, put $P(R) = \infty$.
- A *valuation ring* is an integral domain A with fraction field K, so that for each $x \in K$ either $x \in A$ or $x^{-1} \in A$. Equivalently, there exists a totally ordered abelian group Γ and a valuation $\nu\colon K \to \Gamma \cup \{\infty\}$ so that $A = \{x \in K \mid \nu(x) \geq 0\}$.

Date: December 10, 2022.

*In hermitian K-theory, the split-orthogonal group is often referred to simply as the orthogonal group, and occasionally it is even denoted $O_n(R)$. We follow conventions that agree with classical definitions, so for instance $O_n(\mathbb{R})$ is the usual orthogonal group (viewed as a discrete group).

- A local ring A with residue field $\mathbf{k}$ is *henselian* if it satisfies Hensel's lemma: for any polynomial $f \in A[X]$, if $\alpha \in \mathbf{k}$ has $\bar{f}(\alpha) = 0$ and $\bar{f}'(\alpha) \neq 0$, then there is $a \in A$ with $f(a) = 0$ and $\bar{a} = \alpha$.

Our main result is:

Theorem A. Let A be a valuation ring with $2 \in A^\times$, and denote by $\mathbf{k}$ and K the residue field and the quotient field of A, respectively. Assume that either

(i) A is henselian and $P(\mathbf{k}) < \infty$; or
(ii) $P(K) < \infty$.

Then the Euclidean orthogonal groups $O_n(A)$ satisfy homology stability with constant, abelian, and polynomial coefficients.

More detailed statements of this result are given in Theorems 3.7, 3.8, and 3.10 below. In particular, we answer a question of Cathelineau [Cat07], which asks for the precise stability ranges that can be achieved using Vogtmann's methods. Theorem A also partially answers another question posed by Cathelineau in the same article, which asks for a good class of local rings, with infinite Pythagorean* residue fields, whose Euclidean orthogonal groups have homology stability.

In particular, when F is a field with $P(F) < \infty$, the fact that the groups $O_n(F)$ satisfy homology stability with twisted coefficients is not found in the literature, although it follows immediately from the observation that the arguments of [Vog82] fit into the stability framework of [RWW17].

Stiefel complexes. Concretely, we get our stability results by showing that the *Stiefel complexes* $X(\mathbb{E}^n_A)$ of a valuation ring A are highly-connected, subject to arithmetic conditions on the residue or quotient field of A. Here $X(\mathbb{E}^n_A)$ is the simplicial complex whose simplices are *orthonormal frames* in $\mathbb{E}^n_A$ (see Section 2).

In proving our connectivity estimates, we use various basic results from the theory of quadratic forms over semi-local rings. As this material is non-standard, we give a brief overview of the necessary results in Section 1. The general trend of the theory, however, is that many fundamental results for quadratic forms over fields admit extensions to the semi-local setting.

Computations. By [DV10], the stable homology $\varinjlim_n H_*(G_n; F_n)$ of a finite-degree coefficient system F along a sequence of groups $G_1 \to G_2 \to \dots$ can often be computed in terms of two factors: (1) the functor homology $H_*(\coprod_n BG_n; F)$ and (2) the homology of G_∞ with constant coefficients. In [Dja12], Djament uses this to give various general results about the stable homology of split-orthogonal and Euclidean orthogonal groups in a finite-degree coefficient system. In the Euclidean case, these results can be used to compute the stable homology of the types of coefficient systems that appear, for $R = \mathbb{R}$, in the work of Dupont and Sah on scissors congruences.

Combining Djament's stable calculations with our stability results, it is possible to give various generalizations of computations that, in the real case, appear in scissors congruences. Note, however, that plugging $\mathbb{R}$ into our results does not give sharp enough ranges to carry out the arguments in e.g. [DS90].

We follow Vogtmann in defining the ad hoc invariant m_K for K a field in order to get sharper ranges than those which follow from a finite Pythagoras number; by definition, m_K is the smallest number m (possibly $m = \infty$) such that any *positive-definite* quadratic module (V, q) over K contains a unit vector if $\dim_K V \geq m$, where positive-definite means that $q(v)$ is a sum of squares for each $v \in V$. Pythagorean fields K have $m_K = 1$, finite fields have $m_{\mathbb{F}_q} = 2$, and local or global fields K all have $m_K \leq 4$.

Corollary B. Let K be a field of characteristic zero with $P(K) < \infty$, and let $d \in \mathbb{Z}_{\geq 1}$. Then

$$H_i(O_n(K); \Lambda^d_{\mathbb{Q}}(K^n)) = 0$$

*Recall that a field F is said to be *Pythagorean* if $P(F) = 1$.

for

(a) $i \leq \frac{n-m_K-3d-4}{3}$ if $m_K < \infty$;
(b) $i \leq \frac{n-m_K-2d-2}{2}$ if K is formally real and $m_K < \infty$;
(c) $i \leq \frac{n-3-2P(K)}{2P(K)+1} - d - 1$ if $P(K) < \infty$; and
(d) $i \leq \frac{n-2-P(K)}{P(K)+1} - d - 1$ if K is formally real and $P(K) < \infty$.

To see this, note that $\{\wedge^d_{\mathbb{Q}}(K^n)\}_n$ is a degree d coefficient system, and apply Theorem 3.10 and [Dja12, Thm 4].

For K a field, we let $\mathfrak{o}_n(K)$ denote the adjoint representation of $O_n(K)$; in other words, the underlying K-vector space of $\mathfrak{o}_n(K)$ is the space of skew-symmetric $(n \times n)$-matrices, and $A \in O_n(K)$ acts on $\mathfrak{o}_n(K)$ by sending a skew-symmetric matrix $B \in \mathfrak{o}_n(K)$ to the skew-symmetric matrix ABA^{-1}. Then we have:

Corollary C. Let K be a field of characteristic zero with $P(K) < \infty$. Then

$$H_i(O_n(K); \mathfrak{o}_n(K)) \cong \bigoplus_{s+2t+1=i} H_s(O_\infty(K); \mathbb{Q}) \otimes \Omega^{2t+1}_{K/\mathbb{Q}},$$

where $\Omega^*_{K/\mathbb{Q}}$ is the graded K-algebra of Kähler differentials over $\mathbb{Q}$, for

(a) $i \leq \frac{n-m_K-10}{3}$ if $m_K < \infty$;
(b) $i \leq \frac{n-m_K-6}{2}$ if K is formally real and $m_K < \infty$;
(c) $i \leq \frac{n-3-2P(K)}{2P(K)+1} - 3$ if $P(K) < \infty$; and
(d) $i \leq \frac{n-2-P(K)}{P(K)+1} - 3$ if K is formally real and $P(K) < \infty$.

To see this, note that $\{\mathfrak{o}_n(K)\}_n$ is a degree 2 coefficient system, and apply Theorem 3.10 and [Dja12, Cor 6.6].

As a final corollary:

Corollary D. Let p be an odd prime and let $O_n(\mathbb{Z}_{(p)})$ act on $\mathbb{Z}^n_{(p)}$ in the canonical way. Then

$$H_i(O_n(\mathbb{Z}_{(p)}); \mathbb{Z}^n_{(p)}) = 0 \quad \text{for } i \leq \frac{n-8}{2}.$$

To see this, note that $\{\mathbb{Z}^n_{(p)}\}_n$ is a degree 1 coefficient system, and apply Theorem 3.10 and [Dja12, Thm 3]

Organization of the paper. In Section 1, we give an overview of the results which we import from the theory of quadratic forms over semi-local rings. In Section 2, we show high connectivity for the Stiefel complexes of a valuation ring with nice arithmetic properties. In Section 3, we recall the main result of [RWW17] and show that its conditions are satisfied for the problem studied here, thereby proving our main results. In Appendix A we prove various lemmata about arithmetic properties of local rings which are used in Section 2.

Acknowledgements. I was partially supported by the Danish National Research Foundation through the Copenhagen Centre for Geometry and Topology (DRNF151). An earlier draft of this paper was submitted as my master's thesis at the University of Copenhagen in December 2021. I am deeply grateful to my mentor Nathalie Wahl, who suggested the topic and supervised my thesis. I am also thankful to Kasper Andersen for his thorough (and embarrassingly long) list of corrections and comments on the earlier version, and to Jesper Grodal for unstucking me by suggesting that I limit the scope of my project to hereditary local rings, which led to the current semi-hereditary version.

Contents

1. Quadratic forms over local rings

In other sections of this thesis, we frequently use various facts about quadratic forms over rings. Some of these hold over general rings, and some are true over specifically over local or semilocal rings. Apart from recalling basic notions and establishing notation, the purpose of this section is to give an overview of the results that we use above, e.g. when proving connectivity of Stiefel complexes in Section 2.

The theory of quadratic forms over semilocal rings was pioneered by Roy, Kneser, Knebusch and others in the 60's and 70's. The general trend of this development was that many of the facts that Witt proved for quadratic forms over fields can be generalized to the semilocal setting. A good textbook reference for this theory is Baeza's book [Bae78]. For the Witt–Pfister theory of quadratic forms over fields, we refer to Lam's book [Lam05], but also to [Mea71] which contains a proof of the general Hasse–Minkowski theorem.

1.1. Basic facts and definitions.

Let R be a commutative ring.

Definition. A *quadratic module* over R is a pair (V, q) where V is a finitely-generated projetive R-module and q is a *quadratic form* on V, meaning a function $q\colon V \to R$ that satisfies

(i) $q(ax) = a^2q(x)$ for each $a \in R$, $x \in V$.
(ii) The function $B_q\colon V \times V \to R$ defined by $B_q(x, y) = q(x + y) - q(x) - q(y)$ is bilinear.

We say that (V, q) is *non-singular* if B_q is, i.e. if its tensor-hom adjoint

$$\begin{array}{ccc} V & \xrightarrow{d_q} & V^* \\ \cup & & \cup \\ x & \longmapsto & B_q(x,-) \end{array}$$

is an isomorphism. Otherwise, we say that (V, q) is *singular*.

If S is an R-algebra, then any quadratic module (V, q) over R has an associated quadratic module (V_S, q_S) over S with $V_S = S \otimes_R V$ and

$$q_S(s \otimes x) = s^2q(x),$$
$$B_{q_S}(\sum_i s_i \otimes x_i, \sum_j s'_j \otimes x'_j) = \sum_i \sum_j s_i s'_j B_q(s_i, s_j).$$

Given a prime ideal $\mathfrak{p}$ of R, we let $(V(\mathfrak{p}), q(\mathfrak{p})) = (V_{R/\mathfrak{p}}, q_{R/\mathfrak{p}})$. Elementary commutative algebra gives the following useful proposition:

Proposition 1.1. *A quadratic module (V, q) over R is non-singular if and only if $(V(\mathfrak{m}), q(\mathfrak{m}))$ is non-singular for each maximal ideal $\mathfrak{m}$ of R.*

Example 1.2. Let $a \in R$. The function $R \to R$ given by $x \mapsto ax^2$ is a quadratic form and will be denoted by $\langle a\rangle$. Note that $B_{\langle a\rangle}(x, y) = a(x + y)^2 - ax^2 - ay^2 = 2axy$. If $2 \in R^\times$, then $\langle a\rangle$ is non-singular if and only if $a \in R^\times$. Otherwise $\langle a\rangle$ is always singular.

A *quadratic submodule* of a quadratic module (V, q) is a direct summand $U \subseteq V$ viewed as a quadratic module equipped with the restricted form $q|_U$.

Example 1.3. Let (V, q) be a quadratic module over R. Say that $x, y \in V$ are *orthogonal* (under q) if they are so with respect to B_q, that is if $B_q(x, y) = 0$. If $S \subseteq V$ is any set, put

$$S^\perp = \{x \in V \mid B_q(x, s) = 0 \quad \forall\, s \in S\},$$

i.e. the set of elements of V that are orthogonal to all of S. Then $S^\perp$ is referred to as the *orthogonal complement* of S. If (V,q) is non-singular and $S = U$ is a quadratic submodule of V, then we have a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U^\perp & \longrightarrow & V & \longrightarrow & U^* & \longrightarrow & 0 \\ & & & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ & & & & x & \longmapsto & B_q(x,-)|_U & & \end{array},$$

showing that $U^\perp$ is again a quadratic submodule of V.

Proposition 1.4. *Let (V,q) be a quadratic module over R and let $U \subseteq V$ be a finitely-generated projective submodule. If $(U, q|_U)$ is non-singular, then $V = U \oplus U^\perp$. In particular, U is a quadratic submodule of V.*

Proof. The fact that $U \cap U^\perp = 0$ follows from non-singularity of U. We show that $U + U^\perp = V$. Let x be an arbitrary element of V. Then $B_q(x,-)|_U$ is an element of U^*. Since U is non-singular, we must therefore have $z \in U$ with $B_q(x,y) = B_q(z,y)$ for all $y \in U$. Hence $x - z \in U^\perp$ and $x = z + (x-z) \in U + U^\perp$. □

Non-singular quadratic modules over R form a category $\mathrm{Quad}(R)$ in which a morphism from $(V,q) \to (V',q')$ is a form-preserving R-linear map $f\colon V \to V'$, meaning that $q'(f(v)) = q(v)$ for all $v \in V$. An isomorphism in $\mathrm{Quad}(R)$ is called an *isometry*, and isomorphic quadratic modules are said to be *isometric*. We write

$$O(q) = \mathrm{Aut}_{\mathrm{Quad}(R)}((V,q))$$

for the group of self-isometries of a non-singular quadratic module (V,q).

Example 1.5. Let (V,q) be a quadratic module over R and suppose that $v \in V$ has $q(v) \in R^\times$. The R-linear map

$$\tau_v\colon x \mapsto x - \frac{B_q(x,v)}{2q(v)} v$$

is an element of $O(q)$. Note that $\tau_v^2 = \mathrm{id}$; in fact, τ_v maps v to $-v$ and restricts to the identity on $v^\perp$. Geometrically, the map τ_v thus corresponds to reflecting across the hyperplane $v^\perp$.

A celebrated theorem of Cartan and Dieudonné says that if F is a field with $\mathrm{char}\, F \neq 2$ and (V,q) is a non-singular quadratic space over F, then $O(q)$ is generated by hyperplane reflections. More generally, Klingenberg [Kli61] has shown:

Theorem 1.6 (Cartan–Dieudonné–Klingenberg)**.** *Let A be a local ring with $2 \in A^\times$. For any non-singular quadratic module (V,q) over A, the isometry group $O(q)$ is generated by hyperplane reflections.*

It follows that $H_1(O_n(A); M) = 0$ for any 2-divisible $O_n(A)$-module M.

1.2. **Monoidal structure.** The category $\mathrm{Quad}(R)$ has a symmetric monoidal structure called *orthogonal sum*, which we will denote by $\perp\!\!\!\!\bigcirc$. The quadratic module $(V_1,q_1) \perp\!\!\!\!\bigcirc (V_2,q_2)$ has underlying module $V_1 \oplus V_2$ and form $q_1 \perp\!\!\!\!\bigcirc q_2\colon V_1 \oplus V_2 \to R$ given by

$$(q_1 \perp\!\!\!\!\bigcirc q_2)((x_1,x_2)) = q_1(x_1) + q_2(x_2)$$

for $x_1 \in V_1$ and $x_2 \in V_2$. The unit object of $\mathrm{Quad}(R)$ is the trivial R-module equipped with the zero form. Given maps of quadratic modules $f_i\colon (V_i,q_i) \to (V_i',q_i')$ for $i = 1,2$, the map $f_1 \perp\!\!\!\!\bigcirc f_2$ is simply $f_1 \oplus f_2\colon V_1 \oplus V_2 \to V_1' \oplus V_2'$, which one may check is form-preserving. The required natural transformations in the symmetric monoidal structure on $\mathrm{Quad}(R)$ are simply those associated with the symmetric monoidal structure $\oplus$ on the category of finitely generated projective R-modules; with our definition of $q \perp\!\!\!\!\bigcirc q'$, these maps are all form-preserving.

Recall that for $a \in R$, we denote by $\langle a \rangle$ the quadratic form on R given by $x \mapsto ax^2$. If $a_1, \dots, a_n \in R$, we write $\langle a_1, \dots, a_n \rangle = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$, and if $a = a_1 = \cdots = a_n$, we write $n\langle a \rangle$ for short. More generally, if (V, q) is any quadratic module over R, we write

$$nq = \underbrace{q \perp q \perp \cdots \perp q}_{n \text{ times}}.$$

Example 1.7. For each $n \geq 1$, we define

(Euclidean n-space) $\mathbb{E}_R^n = (R^n, n\langle 1 \rangle)$,

(hyperbolic $2n$-space) $\mathbb{H}_R^{2n} = (R^{2n}, n\langle 1, -1 \rangle)$.

Both of these quadratic modules are non-singular; more generally, if $2 \in R^\times$, then the form $\langle a_1, \dots, a_n \rangle$ is non-singular if and only if $a_1, \dots, a_n \in R^\times$.

It is well-known that if $R = F$ is a field with char $F \neq 2$, then any quadratic space (V, q) over F admits a *diagonalization*, meaning an isometry $(V, q) \cong (F^n, \langle a_1, \dots, a_n \rangle)$ for some $a_1, \dots, a_n \in F$. This fact extends to semilocal rings:

Diagonalization Theorem 1.8. *Suppose R is a semilocal ring in which $2 \in R^\times$. If (R^n, q) is a non-singular quadratic module over R, then there is an isometry $(R^n, q) \cong (R^n, \langle a_1, \dots, a_n \rangle)$ for some $a_1, \dots, a_n \in R^\times$. Equivalently, R^n admits a basis consisting of vectors that are orthogonal under q.*

Proof. See [Bae78, Prop I.3.4]. □

If R is a local ring, then every projective R-module is free, so the theorem says that every quadratic module admits a diagonalization.

To prove homogeneity, we use Roy's [Roy68] generalization of Witt's cancellation theorem to quadratic forms over semilocal rings:

Cancellation Theorem 1.9 (Roy)**.** *Let R be a semilocal ring with $2 \in R^\times$ and let (V_1, q_1), (V_2, q_2) and (W, q) be non-singular quadratic modules over R. If*

$$(V_1, q_1) \perp (W, q) \cong (V_2, q_2) \perp (W, q),$$

then

$$(V_1, q_1) \cong (V_2, q_2).$$

Proof. See [Bae78, Cor III.4.3], or [Roy68, Thm 8.1] for the original proof. □

1.3. **A representation theorem.** Let R be a commutative ring.

Definition. Given a quadratic module (V, q) over R and an element $a \in R$, say that (V, q) *represents* a if there is $x \in V$ with

$$q(x) = a.$$

Furthermore, if x can be chosen to lie outside of $\mathfrak{m}V$ for each maximal ideal $\mathfrak{m} \subset R$, then q is said to *primitively represent* a.

Remark. In K-theory and homology stability literature, it is common to consider *unimodular* elements of R^n, i.e. elements $x = (x_1, \dots, x_n) \in R^n$ so that $Rx_1 + \cdots + Rx_n = R$. On the other hand, say that $x \in R^n$ is *primitive* if it satisfies the condition appearing in the previous definition; that is, $x \notin \mathfrak{m}R^n$ for each maximal ideal $\mathfrak{m}$ in R. For valuation rings, these notions coincide. Indeed, if R is any ring, then clearly a unimodular vector $x \in R^n$ must also be primitive; for if $x \in \mathfrak{m}R^n$, then $Rx_1 + \cdots + Rx_n \subseteq \mathfrak{m}$. Suppose now that A is the valuation ring of a valuation $\nu \colon K \to \Gamma \cup \{\infty\}$. Recall that any finitely-generated ideal in A is principal; indeed, if $x_1, \dots, x_n \in A$, then picking x_i with $\nu(x_i) \leq \nu(x_j)$ for each j, we have $Ax_1 + \cdots + Ax_n = Ax_i$ since $x_j = x_j x_i^{-1} x_i$ (we may assume that $x_1, \dots, x_n$ are not all zero, and so minimality implies $x_i \neq 0$) and $\nu(x_j x_i^{-1}) = \nu(x_j) - \nu(x_i) \geq 0$ so $x_j x_i^{-1} \in A$. Thus if $x = (x_1, \dots, x_n) \in A^n$ is primitive, then $Ax_1 + \cdots + Ax_n = Ax_i$ for some i, and $x_i \notin \mathfrak{m}$ since x is primitive, so we conclude that $Ax_1 + \cdots + Ax_n = A$ as desired. In fact, the proof shows more generally that for any ring R

in which every finitely-generated maximal ideal is principal, primitive and unimodular elements coincide.

Classically, one has been interested in determining whether a given quadratic module (V, q) represents a given ring element $a \in R$. We will be interested specifically in guaranteeing that certain quadratic modules represent the multiplicative identity $1 \in R$. Just as for quadratic modules over fields, it is convenient to rephrase such problems as questions about whether certain associated quadratic modules represent zero in a non-trivial way.

Definition. Let (V, q) be a quadratic module over R. Then (V, q) is

(1) *isotropic* if it primitively represents zero,
(2) *universal* if it represents every unit in R.

Hence a quadratic space (V, q) over a field F is isotropic if and only if there is $0 \neq x \in V$ with $q(x) = 0$. Note also that if (V, q) is a quadratic module over a commutative ring R which represents a unit $a \in R^\times$, then a is automatically primitively represented.

We will frequently use the following theorem:

Representation Theorem 1.10. *Let R be a semilocal ring and let (V, q) be a non-singular quadratic module over R. Then (V, q) represents $a \in R^\times$ if and only if $q \perp \langle -a \rangle$ is isotropic.*

The proof depends on a transversality theorem which we now state.

Definition. Let R be a commutative ring and let (V, q) be a quadratic module over R which admits a decomposition

$$(V, q) = (V_1, q_1) \perp \cdots \perp (V_n, q_n) \tag{1.1}$$

Let $x \in V$ and write

$$x = x_1 + \cdots + x_n$$

for the unique representation with $x_i \in V_i$ for each i. Then x is called *transversal* to the decomposition (1.1) if $q(x_i) \in R^\times$ for each i.

Transversality Theorem 1.11. *Let R be a semilocal ring and let (V, q) be a non-singular quadratic module over R which admits a decomposition*

$$(V, q) = (V_1, q_1) \perp \cdots \perp (V_n, q_n).$$

Suppose (V, q) is isotropic and n is even. Then there exists $x \in V$ which is transversal to the decomposition and has $q(x) = 0$.

Proof. See [Bae78, Thm III.5.2]. □

Proof of the representation theorem. If (V, q) represents $a \in R^\times$, pick $v \in V$ with $q(v) = a$. Then $(q \perp \langle -a \rangle)(v, 1) = 0$ and $(v, 1)$ is clearly primitive. For the other direction, it follows from the transversality theorem that we can pick $v \in V$, $x \in R^\times$ with $0 = (q \perp \langle -a \rangle)(v, x) = q(v) - ax^2$. Then v/x has $q(v/x) = q(v)/x^2 = a$ as desired. □

2. Stiefel complexes over local rings

Let R be a commutative ring. Following [Vog82], we consider:

Definition. Let (V, q) be a quadratic module over R.

(i) The *Stiefel complex* $X(q)$ is the simplicial complex whose vertices are the *unit vectors* of (V, q), meaning elements $v \in V$ with $q(v) = 1$, and such that the $v_1, \ldots, v_k$ span a simplex $[v_1, \ldots, v_k]$ of $X(q)$, also called a *frame*, if $B_q(v_i, v_j) = 0$ for $i \neq j$.
(ii) For $k \in \mathbb{Z}_{\geq 1}$, we let $X_k(q)$ denote the poset of frames $[v_1, \ldots, v_l]$ of length $l \leq k$ under the relation $[v_1, \ldots, v_l] \leq [w_1, \ldots, w_m]$ if $\{v_1, \ldots, v_l\} \subseteq \{w_1, \ldots, w_m\}$.

Note that $|X_k(q)|$ is the barycentric subdivision of $|\operatorname{sk}_{k-1} X(q)|$, so

$$|X_k(q)| \simeq |\operatorname{sk}_{k-1} X(q)|,$$

for $1 \leq k \leq \operatorname{rk} V$.

The goal of this section is to show that if R is valuation ring subject to certain arithmetic conditions, then the complex $|X(n\langle 1\rangle)|$ is highly connected when n is large. Equivalently, we will show that $|X_k(n\langle 1\rangle)|$ is $(k-1)$-spherical for large k when n is large.

Intuitively, showing connectivity of $X(n\langle 1\rangle)$ will depend on having a sufficient supply of unit vectors in the link of any simplex. As in [Vog82], we therefore consider arithmetic invariants that force the existence of such unit vectors. The most obvious of these is:

Definition. Let A be a local ring with residue field $\mathbf{k}$. We define $m_A \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ to be the smallest number m such that if V is a non-singular quadratic submodule of $\mathbb{E}^n_A$ for some n and $\dim_{\mathbf{k}} \overline{V} \geq m$, then V contains a unit vector.

Note that by the diagonalization theorem, a non-singular quadratic module (V, q) over A is isometric to a quadratic submodule of $\mathbb{E}^n_A$ for some n if and only if V is non-singular and admits a diagonalization $\langle d_1, \ldots, d_m\rangle$ in which each d_i is a sum of squares.

Definition. Let A be a local ring with residue field $\mathbf{k}$. We define the *Pythagoras number* $P(A) \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ to be the smallest number p such that if $a \in A$ is a sum of squares, then a is a sum of p squares.

For a field F, finiteness of the Pythagoras number $P(F)$ also guarantees the existence of unit vectors in non-singular subspaces of Euclidean spaces, albeit not as efficiently as finiteness of m_F. The following is [Vog82, Prop 1.5], the proof of which we include for completeness:

Lemma 2.1. *Let F be a field of characteristic $\neq 2$ having $P = P(F) < \infty$, and let $V \subseteq \mathbb{E}^n_F$ be a non-singular subspace of codimension k. If $n > Pk$, then V contains a unit vector.*

Proof. Since V is non-singular, we have by Proposition 1.4 that $\mathbb{E}^n_F \cong V \oplus V^\perp$. Let $\langle a_1, \ldots, a_{n-k}\rangle \cong n\langle 1\rangle|_V$ be a diagonalization of V. By picking a diagonalization of $V^\perp$ also, we get a diagonalization

$$n\langle 1\rangle \cong \langle a_1, \ldots, a_{n-k}, b_1, \ldots, b_k\rangle \tag{2.1}$$

of $\mathbb{E}^n_F$. Each $b_i \neq 0$ is represented by $n\langle 1\rangle$, i.e. is a sum of squares; by assumption each b_i is therefore represented by $P\langle 1\rangle$. Thus for each $1 \leq i \leq k$, we have an isometry

$$P\langle 1\rangle \cong \langle b_i, x_{1,i}, \ldots, x_{P-1,i}\rangle \tag{2.2}$$

for some $x_{1,i}, \ldots, x_{P-1,i} \in F^\times$. Since $n > Pk$, we get from (2.1) and (2.2) that

$$\begin{aligned}
&\langle b_1, \ldots, b_k, a_1, \ldots, a_{n-k}\rangle \\
&\quad\cong n\langle 1\rangle \\
&\quad\cong \langle b_1, \ldots, b_k, x_{1,1}, \ldots, x_{P-1,1}, x_{1,2}, \ldots, x_{P-1,2}, \ldots, x_{1,k}, \ldots, x_{P-1,k}, \underbrace{1, \ldots, 1}_{n-Pk \text{ times}}\rangle,
\end{aligned}$$

so by Witt cancellation

$$\langle a_1, \ldots, a_{n-k}\rangle \cong \langle x_{1,1}, \ldots, x_{P-1,1}, x_{1,2}, \ldots, x_{P-1,2}, \ldots, x_{1,k}, \ldots, x_{P-1,k}, \underbrace{1, \ldots, 1}_{n-Pk \text{ times}}\rangle,$$

and in particular $n\langle 1\rangle|_V \cong \langle a_1, \ldots, a_{n-k}\rangle$ represents one as desired. $\square$

For a field F, note that $P(F)$ is related to m_F by the inequality

$$P(F) \leq m_F. \tag{2.3}$$

If $\operatorname{char} F = 2$, then the proof of the next proposition shows that $P(F) = 1$, and the statement holds trivially. Otherwise, assuming $m_F < \infty$ and that $a \in F$ is a sum of $n > m_F$ squares, pick $w \in \mathbb{E}^n_F$ with $q(w) = a$, where we write $q = n\langle 1\rangle$ for short. Then $w^\perp \subset \mathbb{E}^n_F$ has dimension $n - 1 \geq m_F$, so there is $v \in w^\perp$ with $q(v) = 1$. Diagonalizing $\mathbb{E}^n_F$ starting with $\{v, w\}$, we find that $\mathbb{E}_F^n \cong \langle 1, a, d_1, \dots, d_{n-2}\rangle$. By Witt cancellation, we then have $\mathbb{E}_F^{n-1} \cong \langle a, d_1, \dots, d_{n-2}\rangle$, so a is a sum of $n-1$ squares. Proceeding by induction, we find that a is a sum of m_F squares.

In [Vog82], Vogtmann credits the following result, which generalizes (2.3), to Daniel Shapiro:*

Proposition 2.2. *Suppose there is k so that for each pair of unit vectors $e, f \in \mathbb{E}_F^{k+1}$, the subspace $e^\perp \cap f^\perp$ contains a unit vector. Then $P(F) \leq k-1$.*

Proof. If $F = \mathbb{F}_3$ then $m_{\mathbb{F}_3} = P(\mathbb{F}_3) = 1$ by Proposition A.2. If F has characteristic 2, then $x_1^2 + \cdots + x_n^2 = (x_1 + \cdots + x_n)^2$ so $P(F) = 1$, and hence the inequality holds trivially.

Finally, assume that F has characteristic $\neq 2$ and that $F \neq \mathbb{F}_3$. Let $a \in F$. We claim that we can write $a = x^2 - y^2$ where $x \neq 0$. For $a = 0$ this is obvious, as we can take $x = y = 1$. For $a \neq 0$, we claim that $-a \neq z^2$ for some $z \in F^\times$. As there are at most two z with $z^2 = -a$, this is possible as long as $\#F^\times > 2$, and the latter holds since F is neither $\mathbb{F}_2$ or $\mathbb{F}_3$. Then

$$a = \left(\frac{z + az^{-1}}{2}\right)^2 - \left(\frac{z - az^{-1}}{2}\right)^2$$

as claimed.

Assume $a \in F$ is a sum of $n > k$ squares for some n. As shown above, we may write $a = x^2 - y^2$ with $x \neq 0$. Write $c = a/x^2$ and note that $c = 1 - (x/y)^2$. Since a is a sum of n squares, so is c. Pick $w \in \mathbb{E}_F^n$ with $q(w) = c$, where $q = n\langle 1\rangle$. We view $\mathbb{E}_F^n$ as a subspace of $\mathbb{E}_F^{n+1}$, and let e denote the $(n+1)$st standard basis element of $\mathbb{E}_F^{n+1}$. In particular, $q(e) = 1$. Putting $f = (x/y)e + w$, we then have $q(f) = (x/y)^2 + 1 - (x/y)^2 = 1$.

By assumption $e^\perp \cap f^\perp = e^\perp \cap w^\perp \cap f^\perp$ contains a unit vector v. Diagonalize $\mathbb{E}_F^n$ starting with the vectors e, v, w, so that $\mathbb{E}_F^{n+1} \cong \langle 1, 1, c, d_1, \dots, d_{n-3}\rangle$. By Witt cancellation we thus have $\mathbb{E}_F^{n-1} \cong \langle c, d_1, \dots, d_{n-3}\rangle$. It follows that c is a sum of $n-1$ squares. Proceeding inductively, we find that c is a sum of $k-1$ squares. □

Lemma 2.3. *Let R be a semi-hereditary ring, and let*

$$0 \to K \to E \xrightarrow{f} F$$

be an exact sequence of R-modules, where E and F are projective and E is finitely-generated. Then K is a direct summand of E.

Proof. The image $F' = f(E) \subseteq F$ is finitely-generated, so F' is projective by [Alb61]. Hence

$$0 \to K \to E \xrightarrow{f} F' \to 0$$

splits. □

Recall that semi-hereditary local rings are the same as valuation rings. (Indeed [Kap74, Thm 64] says that an integral domain R is semi-hereditary if and only if $R_\mathfrak{m}$ is a valuation ring for each maximal ideal $\mathfrak{m}$ of R; and [Jen66] shows that a semi-hereditary local ring is integral.)

Lemma 2.4. *Let A be a local ring with maximal ideal $\mathfrak{m}$, and let (V, q) be a quadratic A-module. Then V admits a decomposition*

$$V \cong R \mathbin{\bigcirc\!\!\!\!\perp} W,$$

where W is free and non-singular, and $q(x) \in \mathfrak{m}$ for each $x \in R$.

Proof. Over $\mathbf{k} = A/\mathfrak{m}$, we have $\overline{V} \cong \mathbf{R} \mathbin{\bigcirc\!\!\!\!\perp} \mathbf{W}$, where $\mathbf{R}$ is the radical of $\overline{V}$ and $\mathbf{W}$ is non-singular. By [Bae78, Cor I.3.4], we then have a decomposition $V \cong R \mathbin{\bigcirc\!\!\!\!\perp} W$ where $\overline{W} \cong \mathbf{W}$ and $\overline{R} = \mathbf{R}$. It follows that W is non-singular and $\overline{q(x)} = 0$, i.e. $q(x) \in \mathfrak{m}$, for each $x \in R$. □

Given a quadratic module (V, q) over A, let us write $W \leq V$ when W is a quadratic submodule of V, i.e. W is a direct summand in V. Note that this defines a partial order on the set of submodules of V.

*The proof given by Vogtmann does not go through for fields of characteristic two and $F = \mathbb{F}_3$. It was pointed out to me by Kasper Andersen that these are the only fields for which the proof fails. I give the proof suggested by Andersen here, which fixes the errors in Vogtmann's proof.

Lemma 2.5. *Let A be a valuation ring, and let $[u_1, \dots, u_r]$ and $[v_1, \dots, v_s]$ be possibly empty frames in $\mathbb{E}_A^n$ spanning quadratic submodules U and V respectively, and with $r \geq s \geq 0$. Then $U^\perp \cap V^\perp$ contains a non-singular subspace of dimension at least $n - r - 2s$.*

Proof. By definition, there is an exact sequence

$$0 \to U^\perp \cap V^\perp \to U^\perp \to V^*. \tag{2.4}$$

It follows from Lemma 2.3 that $U^\perp \cap V^\perp \leq U^\perp$. Pick a decomposition

$$U^\perp \cap V^\perp \cong R \circledperp W$$

as in Lemma 2.4. Since $R \leq U^\perp$ and $U^\perp$ is non-singular, there is an $S \leq U^\perp$ which maps isomorphically to $R^* \leq (U^\perp)^*$ under the duality isomorphism

$$\begin{aligned} U^\perp &\xrightarrow{\ \delta\ } (U^\perp)^* \\ x &\longmapsto B_{\langle n \rangle}(x, -). \end{aligned}$$

We claim that $S \cap V^\perp = 0$. Indeed, note that by exactness of

$$0 \to S \cap V^\perp \to S \to V^*$$

and Lemma 2.3, we have that $S \cap V^\perp \leq S$. Hence $\delta(S \cap V^\perp) \leq R^*$. But $S \cap V^\perp \subseteq U^\perp \cap V^\perp$, so by construction of R we have $\delta(S \cap V^\perp) \subseteq \mathfrak{m}R^*$. Since $\delta(S \cap V^\perp)$ is a direct summand of R^* but is contained in $\mathfrak{m}R^*$, we must have $\delta(S \cap V^\perp) = 0$, and hence $S \cap V^\perp = 0$. But then

$$\dim R^* + n - s = \dim S + \dim V^\perp = \dim(S + V^\perp) \leq n,$$

so $\dim R = \dim R^* \leq s$. It follows that $\dim W = \dim(U^\perp \cap V^\perp) - \dim R \geq n - r - 2s$, proving the lemma. □

Proposition 2.6. *Let A be a valuation ring with residue field $\mathbf{k}$. Let $[u_1, \dots, u_r]$ and $[v_1, \dots, v_s]$ be possibly empty frames in $\mathbb{E}_A^n$ spanning quadratic submodules U and V respectively, and with $r \geq s \geq 0$. Then $U^\perp \cap V^\perp$ contains a unit vector if*

- *(i) $n \geq m_A + r + 2s$; or*
- *(ii) $n \geq m_A + r + s$ and $\mathbf{k}$ is formally real;*
- *(iii) $n > 2P(\mathbf{k})r + s$ and A is henselian; or*
- *(iv) $n > P(\mathbf{k})r + s$, A is henselian, and $\mathbf{k}$ is formally real.*

Proof. As in the proof of the previous lemma, we find that $U^\perp \cap V^\perp \leq \mathbb{E}_A^n$. If $\mathbf{k}$ is formally real, then $U^\perp \cap V^\perp$ is non-singular and (ii) follows from the definition of m_A. Similarly, (iv) follows from Lemma 2.1 and Lemma A.5.

If $\mathbf{k}$ is not formally real, then Lemma 2.5 guarantees the existence of a non-singular quadratic subspace $W \leq U^\perp \cap V^\perp \leq \mathbb{E}_A^n$ with $\dim W \geq n - r - 2s$. Hence (i) follows from the definition of m_A and (iii) follows from Lemma 2.1 and Lemma A.5, along with the fact that W is a codimension $2s$ subspace of $U^\perp$ and that $U^\perp \cong \mathbb{E}_A^{n-r}$, as one sees by applying Witt cancellation to the decomposition $\mathbb{E}_A^r \circledperp U^\perp \cong U \circledperp U^\perp \cong \mathbb{E}_A^n$. □

Proposition 2.7. *Let A be a valuation ring with $2 \in A^\times$, and denote the quotient field of A by K. Let $[u_1, \dots, u_r]$ and $[v_1, \dots, v_s]$ be possibly empty frames in $\mathbb{E}_A^n$ spanning quadratic submodules U and V respectively, and with $r \geq s \geq 0$. Then $U^\perp \cap V^\perp$ contains a unit vector if*

- *(i) $n \geq m_K + 2r + s$;*
- *(ii) $n \geq m_K + r + s$ and K is formally real;*
- *(iii) $n > 2P(K)r + s$; or*
- *(iv) $n > P(K)r + s$ and K is formally real.*

Proof. The proof is analogous to the proof of the previous proposition, using Lemma A.7 to push unit vectors over K into unit vectors over A. □

Theorem 2.8. *Let A be a valuation ring with $2 \in A^\times$, and denote the residue and quotient fields of A by $\mathbf{k}$ and K, respectively. Let $[u_1, \dots, u_r]$ and $[v_1, \dots, v_s]$ be possibly empty frames in $\mathbb{E}^n_A$ spanning quadratic submodules U and V respectively, and with $r \geq s \geq 0$. Then*

$$|X_l(U^\perp \cap V^\perp)| \simeq \bigvee S^{l-1}$$

if either

(i) A is henselian and $n \geq 2(r+l-1) + (s+l-1) + m_A$;
(ii) A is henselian and $n \geq 2P(\mathbf{k})(r+l-1) + (s+l-1) + 1$;
(iii) A is henselian, $\mathbf{k}$ is formally real, and $n \geq (r+l-1) + (s+l-1) + m_A$;
(iv) A is henselian, $\mathbf{k}$ is formally real, and $n \geq P(\mathbf{k})(r+l-1) + (s+l-1) + 1$;
(v) $n \geq 2(r+l-1) + (s+l-1) + m_K$;
(vi) $n \geq 2P(K)(r+l-1) + (s+l-1) + 1$;
(vii) K is formally real and $n \geq (r+l-1) + (s+l-1) + m_K$; or
(viii) K is formally real and $n \geq P(K)(r+l-1) + (s+l-1) + 1$.

Corollary 2.9. *Let A be a valuation ring with $2 \notin A^\times$, and denote the residue field of A by $\mathbf{k}$. The Stiefel complex $X(n\langle 1\rangle)$ is*

(i) $\left(\frac{n-m_A-3}{3}\right)$-connected if $m_A < \infty$;
(ii) $\left(\frac{n-5-2P(\mathbf{k})}{2P(\mathbf{k})+1}\right)$-connected if $P(\mathbf{k}) < \infty$;
(iii) $\left(\frac{n-m_A-2}{2}\right)$-connected if $m_A < \infty$; and
(iv) $\left(\frac{n-4-P(\mathbf{k})}{P(\mathbf{k})+1}\right)$-connected if A is henselian and $P(\mathbf{k}) < \infty$.

Further, if K denotes the quotient field of A, then the Stiefel complex $X(n\langle 1\rangle)$ is

(v) $\left(\frac{n-m_K-3}{3}\right)$-connected if $m_K < \infty$;
(vi) $\left(\frac{n-5-2P(K)}{2P(K)+1}\right)$-connected if $P(K) < \infty$;
(vii) $\left(\frac{n-m_A-2}{2}\right)$-connected if K is formally real and $m_K < \infty$; and
(viii) $\left(\frac{n-4-P(\mathbf{k})}{P(\mathbf{k})+1}\right)$-connected if K is formally real and $P(K) < \infty$

The proof uses three lemmata from the homotopy theory of posets:

Lemma 2.10 (Discrete Morse theory)**.** *Let X be a poset with $X = X_0 \cup L_1 \cup \cdots \cup L_n$ as sets. Put $X_j = X_0 \cup L_1 \cup \cdots \cup L_j$ for each $1 \leq j \leq n$. Assume there is $d \in \mathbb{Z}_{\geq 1}$ such that*

(i) $|X_0| \simeq \bigvee S^d$.
(ii) Distinct elements of L_i are not comparable for each $i \geq 1$.
(iii) For each $i \geq 1$ and $x \in L_i$,

$$|\operatorname{Lk}_X(x) \cap X_{i-1}| \simeq \bigvee S^{d-1}.$$

Then $|X| \simeq \bigvee S^d$.

Proof. By induction, it suffices to consider the case $n = 1$. Using Zorn's lemma, pick a maximal subset $L' \subseteq L_1$ with the property that $|X_0 \cup L'| \simeq \bigvee S^d$. We claim that $L' = L_1$. Otherwise pick $x \in L_1 \setminus L'$. Then

$$\begin{aligned} |X_0 \cup L' \cup \{x\}| &= |X_0 \cup L'| \cup_{|\operatorname{Lk}_X(x) \cap (X_0 \cup L')|} |\operatorname{St}_X(x) \cap (X_0 \cup L')| \\ &= |X_0 \cup L'| \cup_{|\operatorname{Lk}_X(x) \cap X_0|} |\operatorname{St}_{X_0 \cup \{x\}}(x)| \end{aligned}$$

using that x is not comparable to any element of L'. Since $|X_0 \cup L'| \simeq \bigvee S^d$, the attaching map $\bigvee S^{d-1} \simeq |\operatorname{Lk}_X(x) \cap X_0| \hookrightarrow |X_0 \cup L'|$ is nulhomotopic. Hence

$$\begin{aligned} |X_0 \cup L' \cup \{x\}| &\simeq \left(\bigvee S^d\right) \vee \left(C|\operatorname{Lk}_X(x) \cap X_0| \cup_{|\operatorname{Lk}_X(x) \cap X_0|} |\operatorname{St}_{X_0 \cup \{x\}}(x)|\right) \\ &\simeq \left(\bigvee S^d\right) \vee \Sigma|\operatorname{Lk}_X(x) \cap X_0| \\ &\simeq \bigvee S^d, \end{aligned}$$

contradicting the maximality of L'. □

Remark. The statement also holds (with exactly the same proof) when we allow $\bigvee S^k$ to mean the empty wedge of k-spheres, i.e. the point $*$.

Lemma 2.11 (Poset deformation lemma)**.** *Let X be a poset. If $f\colon X \to X$ is a poset endomorphism with $f(x) \leq x$ for each $x \in X$, then $|f|\colon |X| \to |\operatorname{im} f|$ is a homotopy equivalence.*

Lemma 2.12. *Let X be a poset. If $X = Y \sqcup Z$ and $y \leq z$ for each $y \in Y$, $z \in Z$, then $|X| \simeq |Y| * |Z|$.*

Proof of Theorem 2.8. In order to deal with the cases (i)-(viii) simultaneously, we let

$$g\colon \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 1}$$

denote the function given by $g(l, r, s) = 2(r + l - 1) + (s + l - 1) + m_A$ in case (i), by $g(l, r, s) = 2P(\mathbf{k})(r + l - 1) + (s + l - 1) + 1$ in case (ii), and so forth, and assume that A satisfies the requirements of the corresponding case.

The proof is by induction on l. For $l = 1$, Proposition 2.6 or 2.7 implies that if $n \geq g(1, r, s)$, then $X_1(U^\perp \cap V^\perp)$ contains a unit vector u. But then $u \neq -u \in X_1(U^\perp \cap V^\perp)$ also, so the discrete poset $X_1(U^\perp \cap V^\perp)$ has at least two elements, and hence $|X_1(U^\perp \cap V^\perp)| \simeq \bigvee S^0$ as desired.

Assume now that $l \geq 2$ and $n \geq g(l, r, s)$. Write $X = X_l(U^\perp \cap V^\perp)$ for convenience. Since g is monotonic in each variable, we have in particular

$$g(l, r, s) \geq g(1, r, s),$$

so as above we find by Proposition 2.6 or 2.7 that we can pick a unit vector $u \in U^\perp \cap V^\perp$. Set $W = U^\perp \cap V^\perp \cap u^\perp$. Since

$$g(l, r, s) \geq g(l - 1, r, s + 1),$$

the induction hypothesis gives $|X_{l-1}(W)| \simeq \bigvee S^{l-2}$.

Write the poset X as a union

$$X = X_0 \cup L_1 \cup \cdots \cup L_l$$

of sets, where

$$\begin{aligned}
X_0 &= \{[\pm u]\} \cup \{[v_1, \ldots, v_i] \mid \exists\, t \geq 1,\, [v_1, \ldots, v_t] \in X_{l-1}(W) \,\wedge\, B_q(v_j, u) \neq 0 \text{ for } t < j \leq i\},\\
L_1 &= \{[v_1, \ldots, v_l] \in X \mid v_1, \ldots, v_l \in U^\perp \cap V^\perp \cap u^\perp\}, \quad \text{and}\\
L_i &= \{[v_1, \ldots, v_i] \mid B_q(v_j, u) \neq 0 \text{ for all } j\}, \quad 2 \leq i \leq l.
\end{aligned}$$

We verify the conditions of the discrete Morse theory lemma with $d = l - 1$. For each $i \geq 1$, all frames in L_i have the same length and thus no distinct elements of L_i are comparable. It remains to check conditions (i) and (iii) of the lemma.

Claim 1. $|X_0| \simeq \bigvee S^d$.

Proof of claim. Put

$$X_0' = \{[\pm u]\} \cup \{[\pm u, v_1, \ldots, v_i] \in X\} \cup \{[v_1, \ldots, v_i] \in X \mid B_q(v_j, u) = 0 \text{ for all } j\}.$$

Then $|X_0'| = \Sigma |X_{l-1}(W)| \simeq \bigvee S^{l-1}$. We define a poset map $f\colon X_0 \to X_0$ which deforms X_0 onto X_0' as in the poset deformation lemma. Set f to be the identity on X_0'. Suppose $[v_1, \ldots, v_i] \in X$ and t are such that $[v_1, \ldots, v_t] \in X_{l-1}(W)$ and $B_q(v_j, u) \neq 0$ for $t < j \leq i$. Then we define $f([v_1, \ldots, v_i]) = [v_1, \ldots, v_t]$ and $f([\pm u, v_1, \ldots, v_i]) = [\pm u, v_1, \ldots, v_t]$, and f is as desired. ■

As in the Morse theory lemma, we put $X_j = X_0 \cup L_1 \cup \cdots \cup L_j$ for each $1 \leq j \leq l$. We must check:

Claim 2. $|\operatorname{Lk}_X([v_1, \ldots, v_i]) \cap X_{i-1}| \simeq \bigvee S^{l-2}$ *for all $i \geq 1$ and $[v_1, \ldots, v_i] \in L_i$.*

Proof of claim. Suppose first that $[v_1, \ldots, v_l] \in L_1$. Then

$$\mathrm{Lk}_X([v_1, \ldots, v_l]) \cap X_0 = \mathrm{Lk}_X([v_1, \ldots, v_l]) = \{\text{proper subframes of } [v_1, \ldots, v_l]\}.$$

This can be identified with the barycentric subdivision of the boundary of an $(l-1)$-simplex, so $|\mathrm{Lk}_X([v_1, \ldots, v_l]) \cap X_0| \simeq \bigvee S^{l-2}$ as desired.

Now suppose that $[v_1, \ldots, v_i] \in L_i$ for some $i \geq 2$. Then

$$\begin{aligned}
&\mathrm{Lk}_X([v_1, \ldots, v_i]) \cap X_{i-1} \\
&\quad = \{\text{proper subframes of } [v_1, \ldots, v_i]\} \\
&\quad\quad \cup \{[w_1, \ldots, w_t, v_1, \ldots, v_i] \mid 1 \leq t \leq l-i,\ [w_1, \ldots, w_t] \in X_{l-1}(W)\} \\
&\quad = \{\text{proper subframes of } [v_1, \ldots, v_i]\} \\
&\quad\quad \cup \{[w_1, \ldots, w_t, v_1, \ldots, v_i] \mid [w_1, \ldots, w_t] \in X_{l-i}(W \cap [v_1, \ldots, v_i]^\perp)\}.
\end{aligned}$$

Thus Lemma 2.12 implies that

$$\begin{aligned}
&|\mathrm{Lk}_X([v_1, \ldots, v_i]) \cap X_{i-1}| \\
&\quad \simeq |\{\text{proper subframes of } [v_1, \ldots, v_i]\}| \\
&\quad\quad * |\{[w_1, \ldots, w_t, v_1, \ldots, v_i] \mid [w_1, \ldots, w_t] \in X_{l-i}(W \cap [v_1, \ldots, v_i]^\perp)\}| \\
&\quad \simeq S^{i-2} * |X_{l-i}(W \cap [v_1, \ldots, v_i]^\perp)| \\
&\quad \simeq S^{i-2} * S^{l-i-1} \simeq S^{l-2},
\end{aligned}$$

where we have used that

$$g(l, r, s) \geq g(l-i, r, s+i),$$

so the induction hypothesis applies to $X_{l-i}(W \cap [v_1, \ldots, v_i]^\perp)$. ■

It follows from the discrete Morse theory lemma that $|X|$ is homotopy equivalent to a wedge of $(l-1)$-spheres, thus completing the induction. □

3. Homology stability of $O_n(A)$

For a pair of objects (A, X) in a *homogeneous* monoidal category $(\mathcal{C}, \oplus, 0)$, the main result of [RWW17] says that the sequence of groups

$$\mathrm{Aut}_{\mathcal{C}}(A) \xrightarrow{-\oplus X} \mathrm{Aut}_{\mathcal{C}}(A \oplus X) \xrightarrow{-\oplus X} \ldots \xrightarrow{-\oplus X} \mathrm{Aut}_{\mathcal{C}}(A \oplus X^{\oplus n}) \xrightarrow{-\oplus X} \ldots$$

satisfies homology stability if a certain associated semi-simplicial set $W_n(A, X)_\bullet$ is sufficiently connected for large n. In this section, we derive homology stability of the Euclidean orthogonal groups $O_n(A)$ for some local rings by applying this framework to the pair $(0, \mathbb{E}^1_A)$ in the category $\mathrm{Quad}(A)$ of non-singular quadratic modules over A.

3.1. **Homogeneity of** $\mathrm{Quad}(R)$**.** In order to apply this framework, we first show that the category $\mathrm{Quad}(A)$ is homogeneous. Recall that for a monoidal category $(\mathcal{C}, \oplus, 0)$ in which the monoidal unit 0 is initial, this means that for all objects $A, B \in \mathcal{C}$:

- **H1** The action of $\mathrm{Aut}_{\mathcal{C}}(B)$ on the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ by postcomposition is transitive.
- **H2** The group homomorphism

$$\begin{aligned}
\mathrm{Aut}_{\mathcal{C}}(A) &\longrightarrow \mathrm{Aut}_{\mathcal{C}}(A \oplus B) \\
f &\longmapsto f \oplus B
\end{aligned}$$

  is injective, with image $\mathrm{Fix}(0 \oplus B) = \{\varphi \in \mathrm{Aut}_{\mathcal{C}}(A \oplus B) \mid \varphi \circ \iota_B = \iota_B\}$, where ι_B is the canonical morphism $B \cong 0 \oplus B \xrightarrow{0_A \oplus B} A \oplus B$.

Before proving homogeneity, we make the basic observation:

Proposition 3.1. *Let R be a commutative ring and let $f\colon (V, q_V) \to (U, q_U)$ be a map of quadratic modules over R. Then for each $x, y \in V$, $B_{q_U}(f(x), f(y)) = B_{q_V}(x, y)$.*

Proof. By definition

$$\begin{aligned} B_{q_U}(f(x),f(y)) &= q_U(f(x)+f(y)) - q_U(f(x)) - q_U(f(y)) \\ &= q_V(x+y) - q_V(x) - q_V(y) \\ &= B_{q_V}(x,y). \end{aligned}$$

□

Corollary 3.2. *Let R be a commutative ring and let $f\colon (V,q_V) \to (U,q_U)$ be a map of quadratic modules over R. If the quadratic module (V,q_V) is non-singular, then f is injective.*

Proof. Let $0 \neq x \in V$. By assumption, the duality map $d_{q_V}: V \to V^*$ is an isomorphism; in particular, it is injective. Hence $B_{q_V}(x,-) \neq 0$, i.e. there is $y \in V$ with $B_{q_V}(x,y) \neq 0$. By the lemma, we then have $B_{q_U}(f(x),f(y)) = B_{q_V}(x,y) \neq 0$, so $f(x) \neq 0$ as desired. □

Theorem 3.3. *Let R be a semilocal ring with $2 \in R^\times$. Then the symmetric monoidal category $(\mathrm{Quad}(R), \perp, 0)$ of non-singular quadratic modules is homogenenous.*

Proof. First observe that the trivial quadratic module 0 is initial in $\mathrm{Quad}(R)$. Condition H2 is clear from the definition of $\perp$; indeed, given non-singular quadratic modules (V,q_V) and (W,q_W), $\mathrm{Fix}(0 \perp W)$ is just the set of automorphisms $\varphi \in O(V \perp W)$ which fix $0 \perp W$ pointwise. If φ is such a morphism, we must then have $\varphi(V \perp 0) = (0 \perp W)^\perp = V \perp 0$; i.e. φ must map the V-summand to itself, and letting φ_V be the automorphism of V thus defined, we find that φ is the image of φ_V under the map $O(V) \to O(V \perp W)$.

To verify condition H1, let (V,q_V) and (W,q_W) be non-singular quadratic modules, and let $f,g\colon (V,q_V) \to (W,q_W)$ be two maps between them. We note that f and g map (V,q_V) isometrically onto non-singular quadratic submodules $f(V)$ and $g(V) \subseteq W$ respectively (cf. Proposition 1.4). For this, it suffices to note that f and g are injective by Corollary 3.2.

Hence Proposition 1.4 gives that

$$(W,q_W) = (f(V),q_W) \perp (f(V)^\perp, q_W) \cong (V,q_W) \perp (f(V)^\perp, q_W)$$

and similarly for $g(V)$, so

$$(V,q_W) \perp (f(V)^\perp, q_W) \cong (V,q_W) \perp (g(V)^\perp, q_W).$$

By the cancellation theorem, we thus have an isometry $h\colon (f(V)^\perp, q_W) \to (g(V)^\perp, q_W)$. But then $f = (fg^{-1} \perp h) \circ g$ and $fg^{-1} \perp h \in O(q_W)$ as desired. □

3.2. **The space of destabilizations.** In this subsection, we identify the space of destabilizations $|W_n(0,\mathbb{E}^1_A)_\bullet|$. First recall the following constructions [RWW17]:

Definition. Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category in which the unit object 0 is initial.

Let A and X be objects in $\mathcal{C}$. For each $n \geq 1$, define

(i) a semi-simplicial set $W_n(A,X)_\bullet$ having

$$W_n(A,X)_p = \mathrm{Hom}_{\mathcal{C}}(X^{\oplus(p+1)}, A \oplus X^{\oplus n})$$

for each $p \geq 0$, with face maps

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X^{\oplus(p+1)}, A \oplus X^{\oplus n}) &\xrightarrow{\ d_i\ } \mathrm{Hom}_{\mathcal{C}}(X^{\oplus p}, A \oplus X^{\oplus n}) \\ f &\longmapsto f \circ (X^{\oplus i} \oplus (0 \to X) \oplus X^{\oplus p-i}) \end{aligned}$$

for each $0 \leq i \leq p$; and

(ii) a simplicial complex $S_n(A,X)$ with set of vertices $W_n(A,X)_0$, and such that vertices $f_0,\dots,f_p \in W_n(A,X)_0$ spans a simplex of $S_n(A,X)$ if and only if there is a simplex $\sigma \in W_n(A,X)_p$ whose vertices are exactly $\{f_0,\dots,f_p\}$.

Intuitively, $|W_n(A,X)_\bullet|$ parametrizes ways of cutting out copies of sums of X from $A \oplus X^{\oplus n}$, which is why [RWW17] refer to it as the n-th *space of destabilizations* of A by X.

We now identify the objects $W_n(0,\mathbb{E}^1_A)_\bullet$ and $S_n(0,\mathbb{E}^1_A)$.

Definition. Let (V, q) be a quadratic space over a commutative ring R. The *ordered Stiefel space* $\vec{X}(q)_\bullet$ is the semi-simplicial set having

$$\vec{X}(q)_p = \{(v_0, \dots, v_p) \in V^{p+1} \mid q(v_i) = 1 \text{ for each } i \text{ and } B_q(v_i, v_j) = 0 \text{ for } i \neq j\}$$

for each $p \geq 0$, with face maps

$$\begin{aligned} \vec{X}(q)_p &\xrightarrow{\quad d_i \quad} \vec{X}(q)_{p-1} \\ (v_0, \dots, v_p) &\longmapsto (v_0, \dots, \widehat{v_i}, \dots, v_p) \end{aligned}$$

for each $0 \leq i \leq p$.

An element $(v_0, \dots, v_p) \in \vec{X}(q)_p$ is referred to as an *ordered frame* in (V, q).

Proposition 3.4. *Let R be a commutative ring and consider the symmetric monoidal category $(\mathrm{Quad}(R), \perp, 0)$. For any object $(V, q) \in \mathrm{Quad}(R)$ and each $n \geq 1$, there is an isomorphism of semi-simplicial sets*

$$W_n((V, q), \mathbb{E}^1_R)_\bullet \cong \vec{X}((V, q) \perp \mathbb{E}^n_R)_\bullet$$

given on p-simplices by

$$f \mapsto (f(e_1), \dots, f(e_{p+1})) \tag{3.1}$$

for $0 \leq i \leq p$, where e_i denotes the unit vector in $\mathbb{E}^{p+1}_R$ having i-th coordinate equal to 1 and all other coordinates zero.

Proof. Note that $(f(e_1), \dots, f(e_{p+1}))$ is indeed an ordered frame by Proposition 3.1, so the map is well-defined. To see that it is an isomorphism of semi-simplicial sets, note that for each p, the map (3.1) has an inverse $\vec{X}((V, q) \perp \mathbb{E}^n_A)_p \to W_n((V, q), \mathbb{E}^1_R)_p$ given by sending $(v_0, \dots, v_p)$ to the uniquely-determined R-linear map $f\colon R^{p+1} \to V \oplus R^n$ which maps e_i to v_{i-1}. This map is form-preserving, since for each $\sum_{i=1}^{p+1} r_i e_i$ we have

$$\begin{aligned} (q \perp n\langle 1\rangle)\left(f\left(\sum_{i=1}^{p+1} r_i e_i\right)\right) &= (q \perp n\langle 1\rangle)\left(\sum_{i=1}^{p+1} r_i v_{i-1}\right) \\ &= \sum_{i=1}^{p+1} (q \perp n\langle 1\rangle)(r_i v_{i-1}) + \sum_{1 \leq i < j \leq p+1} r_i r_j B_{q \perp n\langle 1\rangle}(v_{i-1}, v_{j-1}) \\ &= r_1^2 + \cdots + r_{p+1}^2 = (p+1)\langle 1\rangle \left(\sum_{i=1}^{p+1} r_i e_i\right). \end{aligned}$$

□

Corollary 3.5. *There is an isomorphism of simplicial complexes*

$$S_n((V, q), \mathbb{E}^1_R) \cong X(q \perp n\langle 1\rangle)$$

given by $f \mapsto f(e_1) = f(1)$.

3.3. **Homology stability.** As mentioned at the beginning of this section, the main theorem of Randal-Williams–Wahl [RWW17] requires knowing that the space of destabilizations $|W_n(A, X)_\bullet|$ is highly-connected. Under certain conditions, Randal-Williams and Wahl show that connectivity estimates for $|S_n(A, X)|$ imply connectivity estimates for $|W_n(A, X)_\bullet|$. Specificially, Prop 2.9 and Thm 2.10 in [RWW17] imply that if $(\mathcal{C}, \oplus, 0)$ is a symmetric monoidal homogeneous category which is *locally standard* at a pair of objects (A, X), then for positive integers $a, k \geq 1$, the space of destabilizations $|W_n(A, X)_\bullet|$ is $\left(\frac{n-a}{k}\right)$-connected for all $n \geq 0$ if and only if $|S_n(A, X)|$ is $\left(\frac{n-a}{k}\right)$-connected for all $n \geq 0$. Here being locally standard at (A, X) means that

LS1 the morphisms $(0 \to A) \oplus X \oplus (0 \to X)$ and $(0 \to A \oplus X) \oplus X$ are distinct in $\mathrm{Hom}_{\mathcal{C}}(X, A \oplus X^{\oplus 2})$, and

LS2 for all $n \geq 1$, the map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X, A \oplus X^{\oplus n-1}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X, A \oplus X^{\oplus n}) \\ f & \longmapsto & f \oplus (0 \to X) \end{array}$$

is injective.

Note that

Proposition 3.6. *If R is a semi-local ring with $2 \in R^\times$, then* $\mathrm{Quad}(R)$ *is locally standard at (V, W) for any pair of quadratic modules V and W as long as $V \neq 0$.*

Proof. To see LS1, simply note that the two morphisms are given by $v \mapsto (0, v, 0)$ and $v \mapsto (0, 0, v)$ respectively. Condition LS2 is clear from the definition of $\textcircled{\perp}$. $\square$

Thus we have:

Theorem 3.7. *Let A be a valuation ring with residue field $\mathbf{k}$, and assume that $2 \in A^\times$. Consider the map*

$$H_i(O_n(A); \mathbb{Z}) \to H_i(O_{n+1}(A); \mathbb{Z}) \tag{3.2}$$

induced by

$$\begin{array}{ccc} O_n(A) & \longrightarrow & O_{n+1}(A) \\ f & \longmapsto & f \textcircled{\perp} id. \end{array}.$$

Then (3.2) *is*

- *(i) a surjection for $i \leq \frac{n-m_A-1}{3}$ and an isomorphism for $i \leq \frac{n-m_A-2}{3}$ if $m_A < \infty$; and*
- *(ii) a surjection for $i \leq \frac{n-m_A}{2}$ and an isomorphism for $i \leq \frac{n-m_A-1}{2}$ if $m_A < \infty$ and A is formally real;*
- *(iii) a surjection for $i \leq \frac{n-3-2P(\mathbf{k})}{2P(\mathbf{k})+1}$ and an isomorphism for $i \leq \frac{n-4-2P(\mathbf{k})}{2P(\mathbf{k})+1}$ if A is henselian and $P(\mathbf{k}) < \infty$; and*
- *(iv) a surjection for $i \leq \frac{n-2-P(\mathbf{k})}{P(\mathbf{k})+1}$ and an isomorphism for $i \leq \frac{n-3-P(\mathbf{k})}{P(\mathbf{k})+1}$ if A is henselian, $\mathbf{k}$ is formally real, and $P(\mathbf{k}) < \infty$.*

Further, if K denotes the quotient field of A, then (3.2) *is*

- *(v) a surjection for $i \leq \frac{n-m_K-1}{3}$ and an isomorphism for $i \leq \frac{n-m_K-2}{3}$ if $m_K < \infty$;*
- *(vi) a surjection for $i \leq \frac{n-m_K}{2}$ and an isomorphism for $i \leq \frac{n-m_K-1}{2}$ if K is formally real and $m_K < \infty$;*
- *(vii) a surjection for $i \leq \frac{n-3-2P(K)}{2P(K)+1}$ and an isomorphism for $i \leq \frac{n-4-2P(K)}{2P(K)+1}$ if $P(K) < \infty$; and*
- *(viii) a surjection for $i \leq \frac{n-2-P(K)}{P(K)+1}$ and an isomorphism for $i \leq \frac{n-3-P(K)}{P(K)+1}$ if K is formally real and $P(K) < \infty$.*

Proof. This follows from in [RWW17, Thm 3.1] and Corollary 2.9 above. $\square$

3.4. **Stability with twisted coefficients.** The framework of [RWW17] also gives us homology stability of $O_n(A)$ with certain twisted coefficients, which we now describe.

Recall that a *coefficient system* for the sequence $\{O_n(A)\}_1^\infty$ is a functor $F\colon \mathcal{E} \to \mathcal{A}$, where $\mathcal{A}$ is an abelian category and $\mathcal{E} \subseteq \mathrm{Quad}(A)$ denotes the full subcategory spanned by the Euclidean spaces $\mathbb{E}_A^n$. In particular, functoriality endows $F_n = F(\mathbb{E}_A^n)$ with the structure of a $\mathbb{Z}[O_n(A)]$-module, and the maps $\mathbb{E}_A^n \textcircled{\perp} (0 \to \mathbb{E}_A^1)\colon \mathbb{E}_A^n \to \mathbb{E}_A^{n+1}$ induce maps

$$H_i(O_n(A); F_n) \to H_i(O_{n+1}(A); F_{n+1}).$$

Homology with twisted coefficients is a powerful tool for computing the homology groups in a *family of representations* $\{F_n\}_1^\infty$.

Theorem 3.8. *Let A be a valuation ring with residue field $\mathbf{k}$, and assume that $2 \in A^\times$. Put $O_\infty(A) = \varinjlim O_n(A)$, and suppose M is an $O_\infty(A)$-module on which the commutator subgroup $\Omega_\infty(A) \leq O_\infty(A)$ acts trivially. For each $n \geq 0$, we consider M as an $O_n(A)$-module via restriction of scalars along the canonical map $O_n(A) \to O_\infty(A)$. Consider the map*

$$H_i(O_n(A); M) \to H_i(O_{n+1}(A); M) \tag{3.3}$$

induced by

$$\begin{array}{ccc} O_n(A) & \longrightarrow & O_{n+1}(A) \\ f & \longmapsto & f \perp id \end{array}.$$

Then (3.3) *is*

(i) a surjection for $i \leq \frac{n-m_A-2}{3}$ and an isomorphism for $i \leq \frac{n-m_A-4}{3}$ if $m_A < \infty$;
(ii) a surjection for $i \leq \frac{n-2P(\mathbf{k})-2P(\mathbf{k})-2}{2P(\mathbf{k})+1}$ and an isomorphism for $i \leq \frac{n-2P(\mathbf{k})-2P(\mathbf{k})-4}{2P(\mathbf{k})+1}$ if A is henselian and $P(\mathbf{k}) < \infty$; and
(iii) a surjection for

$$i \leq \tfrac{n-P(\mathbf{k})-\max\{3,P(\mathbf{k})+1\}}{\max\{3,P(\mathbf{k})+1\}}$$

and an isomorphism for

$$i \leq \tfrac{n-P(\mathbf{k})-2-\max\{3,P(\mathbf{k})+1\}}{\max\{3,P(\mathbf{k})+1\}}$$

if $\mathbf{k}$ is formally real, A is henselian, and $P(\mathbf{k}) < \infty$.

Further, if K denotes the quotient field of A, then (3.3) *is*

(iv) a surjection for $i \leq \frac{n-m_K-2}{3}$ and an isomorphism for $i \leq \frac{n-m_K-4}{3}$ if $m_K < \infty$;
(v) a surjection for $i \leq \frac{n-2P(K)-2P(K)-2}{2P(K)+1}$ and an isomorphism for $i \leq \frac{n-2P(K)-2P(K)-4}{2P(K)+1}$ if $P(K) < \infty$; and
(vi) a surjection for

$$i \leq \tfrac{n-P(K)-\max\{3,P(K)+1\}}{\max\{3,P(K)+1\}}$$

and an isomorphism for

$$i \leq \tfrac{n-P(K)-2-\max\{3,P(K)+1\}}{\max\{3,P(K)+1\}}$$

if K is formally real and $P(K) < \infty$.

Example 3.9. For each $n \geq 0$, let $\Omega_n(A) \leq O_n(A)$ denote the commutator subgroup. Then the map $H_i(\Omega_n(A); \mathbb{Z}) \to H_i(\Omega_{n+1}(A); \mathbb{Z})$ is an epimorphism (resp. isomorphism) for the ranges given in the previous theorem (see [RWW17, Sec 3.1]).

We recall what it means for a coefficient system to have *degree r at $N \in \mathbb{Z}$*. The unique map $0 \to \mathbb{E}^1_A$ induces a natural transformation $\sigma\colon \mathrm{id}_{\mathcal{E}} \to - \oplus \mathbb{E}^1_A$. For a coefficient system $F\colon \mathcal{E} \to \mathrm{Mod}_{\mathbb{Z}}$, we put

$$\ker F = \ker \sigma_F \qquad \text{and} \qquad \operatorname{coker} F = \operatorname{coker} \sigma_F,$$

where $\sigma_F = F(\sigma)\colon F \to F \circ (- \oplus \mathbb{E}^1_A)$. Then F is said to have degree $r < 0$ at $N \in \mathbb{Z}$ if $F(\mathbb{E}^n_A) = 0$ for all $n \geq N$, and recursively it is said to have degree $r \geq 0$ at $N \in \mathbb{Z}$ if

(i) The kernel $\ker F$ has degree -1 at N; and
(ii) The cokernel $\operatorname{coker} F$ has degree $r-1$ at $N-1$.

Note that if F and G are coefficient systems of degree r and s at N, then $F \oplus G$ has degree $\max\{r, s\}$ at N.

Theorem 3.10. *Let A be a valuation ring with residue field $\mathbf{k}$, and assume that $2 \in A^\times$. Let $F\colon \mathcal{E} \to Mod_{\mathbb{Z}}$ be a coefficient system of degree r at $N \in \mathbb{Z}$. Consider the map*

$$H_i(O_n(A); F_n) \to H_i(O_{n+1}(A); F_{n+1}) \tag{3.4}$$

induced by

$$\begin{aligned} O_n(A) &\longrightarrow O_{n+1}(A) \\ f &\longmapsto f \textcircled{\perp} id \end{aligned}.$$

Then (3.4) *is*

(i) *a surjection for* $i \leq \frac{n-m_A-1}{3} - r$ *and an isomorphism for* $i \leq \frac{n-m_A-1}{3} - r - 1$ *if* $m_A < \infty$*;*
(ii) *a surjection for* $i \leq \frac{n-m_A}{2} - r$ *and an isomorphism for* $i \leq \frac{n-m_A}{2} - r - 1$ *if* $\mathbf{k}$ *is formally real and* $m_A < \infty$*;*
(iii) *a surjection for* $i \leq \frac{n-3-2P(\mathbf{k})}{2P(\mathbf{k})+1} - r$ *and an isomorphism for* $i \leq \frac{n-3-2P(\mathbf{k})}{2P(\mathbf{k})+1} - r - 1$ *if* A *is henselian and* $P(\mathbf{k}) < \infty$*; and*
(iv) *a surjection for* $i \leq \frac{n-2-P(\mathbf{k})}{P(\mathbf{k})+1} - r$ *and an isomorphism for* $i \leq \frac{n-2-P(\mathbf{k})}{P(\mathbf{k})+1} - r - 1$ *if* A *is henselian,* $\mathbf{k}$ *is formally real and* $P(\mathbf{k}) < \infty$*.*

Further, if K *denotes the quotient field of* A*, then* (3.4) *is*

(v) *a surjection for* $i \leq \frac{n-m_K-1}{3} - r$ *and an isomorphism for* $i \leq \frac{n-m_K-1}{3} - r - 1$ *if* $m_K < \infty$*;*
(vi) *a surjection for* $i \leq \frac{n-m_K}{2} - r$ *and an isomorphism for* $i \leq \frac{n-m_K}{2} - r - 1$ *if* K *is formally real and* $m_K < \infty$*;*
(vii) *a surjection for* $i \leq \frac{n-3-2P(K)}{2P(K)+1} - r$ *and an isomorphism for* $i \leq \frac{n-3-2P(K)}{2P(K)+1} - r - 1$ *if* $P(K) < \infty$*; and*
(viii) *a surjection for* $i \leq \frac{n-2-P(K)}{P(K)+1} - r$ *and an isomorphism for* $i \leq \frac{n-2-P(K)}{P(K)+1} - r - 1$ *if* K *is formally real and* $P(K) < \infty$*.*

3.5. **Homology stability of $O_n(\mathbb{Z})$.** In [Vog82], Vogtmann states that the Euclidean orthogonal groups $O_n(R)$ also display homology stability when R is the ring of integers in a totally real number field K.* In particular, this includes the case $R = \mathbb{Z}$.

Confusingly, Vogtmann seems to treat part of this integral case together with the field case, thus using Witt cancellation and hyperplane reflections to show that $O_n(R)$ acts transitively on p-simplices in the Stiefel complex $X(\mathbb{E}^n_R)$. However, hyperplane reflections do not exist over R (as $2 \notin R^\times$), and Witt cancellation does not hold in general. Instead, transitivity – or, as it corresponds to in the [RWW17] setup, *local* homogeneity of $\mathrm{Quad}(R)$ – can be deduced from [Vog82, Lem 1.3], which states that the only unit vectors in $\mathbb{E}^n_R$ are of the form $\pm e_i$, where $e_1, \ldots, e_n$ denote the standard basis of R^n. Clearly $O_n(R)$ then acts transitively on p-frames.

However, the fact that the unit vectors in $\mathbb{E}^n_R$ admit such a simple description also means that $O_n(R)$ does not really contain any ring-theoretic information. In fact,

Proposition 3.11. *Let* R *be the ring of integers in a totally real number field, and denote by* $\mathrm{Aut}(X(\mathbb{E}^n_R))$ *the group of automorphisms of the simplicial complex* $X(\mathbb{E}^n_R)$*. The homomorphism*

$$O_n(R) \to \mathrm{Aut}(X(\mathbb{E}^n_R)),$$

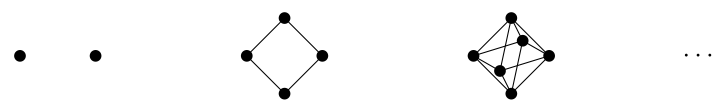
coming from the action of $O_n(R)$ *on* $X(\mathbb{E}^n_R)$*, is an isomorphism of groups.*

Proof. Note that an element $\varphi \in O_n(R)$ is determined by the images of the standard basis $\varphi(e_i)$, $i = 1, \ldots, n$, so injectivity is clear.

For surjectivity, let $f \in \mathrm{Aut}(X(\mathbb{E}^n_R))$. If $\sigma = \{e_1, \ldots, e_n\}$ is the simplex corresponding to the standard basis, then $f(\sigma) = \{f(e_1), \ldots, f(e_n)\}$ is again an n-frame, so there is $\varphi \in O_n(R)$ with $\varphi(e_i) = f(\{e_i\})$. But any automorphism of $X(\mathbb{E}^n_R)$ is uniquely determined by the images of the vertices corresponding to $e_1, \ldots, e_n$, as the vertex corresponding to $-e_i$ can be characterized as the unique vertex in $X(\mathbb{E}^n_R)$ (other than e_i itself) which does not share an edge with e_i. Hence φ must be the desired lift of f. □

*Recall that a field K is *totally real* if for every embedding $\iota\colon K \hookrightarrow \mathbb{C}$, one has $\iota(K) \subseteq \mathbb{R}$.

In other words, ring theory disappears and we are left with combinatorics; the groups $O_n(R)$ are just automorphism groups for the simplicial complexes in the sequence of simplicial spheres



$n = 1$ $\quad$ $n = 2$ $\quad$ $n = 3$

3.6. **K-theoretic interpretation.** As mentioned in the introduction, interest in the split-orthogonal groups $O_{n,n}$ has been driven by its importance in hermitian K-theory. We recall the role of $BO^+_{\infty,\infty}$ in K-theory here and give an analogous interpretation of BO^+_∞.

Let R be a commutative ring.

Proposition 3.12. *The inclusion of the full monoidal subcategory* $\{\mathbb{H}^{2n} \mid n \geq 0\} \subseteq \mathrm{Quad}(R)$ *induces a homotopy equivalence on basepoint components of K-spaces. Hence*

$$K(\mathrm{Quad}(R)) \simeq K_0(\mathrm{Quad}(R)) \times BO_{\infty,\infty}(R)^+,$$

where $O_{\infty,\infty}(R) = \varinjlim_n O_{n,n}(R)$.

Proof. For any projective R-module V, let $\mathbb{H}(V)$ denote the quadratic module $(V \oplus V^*, q_{\mathbb{H}})$ where $q_{\mathbb{H}}(m,\phi) = \phi(m)$. The quadratic module $\mathbb{H}(V)$ is called the *hyperbolic module* associated to M, justified by the fact that $\mathbb{H}(R^n) \cong \mathbb{H}^{2n}$. More generally, one finds by an elementary argument that $(V,q) \circledcirc (V,-q) \cong \mathbb{H}(V)$ for any quadratic module (V,q). If (V,q) is a quadratic module, then since V is finitely-generated projective, there is an isomorphism of modules $R^n \cong V \oplus U$ for some n and U. Thus we find $(V,q) \circledcirc (V,-q) \circledcirc \mathbb{H}(U) \cong \mathbb{H}(V) \circledcirc \mathbb{H}(U) \cong \mathbb{H}(V \oplus U) \cong \mathbb{H}(R^n) \cong \mathbb{H}^{2n}$. This shows that $\{\mathbb{H}^{2n} \mid n \geq 0\}$ is cofinal in $\mathrm{Quad}(R)$, so the inclusion of this monoidal subcategory induces a homotopy equivalence on basepoint components of K-spaces. Since $\{\mathbb{H}^{2n} \mid n \geq 0\}$ is generated by $\mathbb{H}^2$, a standard group completion argument (see [RW13] or [Nik17]) shows that

$$K(\{\mathbb{H}^{2n} \mid n \geq 0\}) \simeq \mathrm{hocolim}\left(\mathcal{H} \xrightarrow{-\circledcirc \mathbb{H}^2} \mathcal{H} \xrightarrow{-\circledcirc \mathbb{H}^2} \dots\right)^+,$$

where $\mathcal{H}$ is the classifying space of the groupoid core of $\{\mathbb{H}^{2n} \mid n \geq 0\}$, i.e. $\mathcal{H} = \coprod_{n \geq 0} BO_{n,n}(R)$. But here

$$\mathrm{hocolim}\left(\mathcal{H} \xrightarrow{-\circledcirc \mathbb{H}^2} \mathcal{H} \xrightarrow{-\circledcirc \mathbb{H}^2} \dots\right) \simeq \mathbb{Z} \times BO_{\infty,\infty}(R),$$

and the conclusion follows. □

Here $K(\mathrm{Quad}(R))$ is the hermitian K-theory space of Karoubi and others, also denoted $\mathcal{L}(R)$. By analogy, the space $BO_\infty(R)^+$ is related to "positive-definite quadratic K-theory".

Proposition 3.13. *Let* $\mathrm{Quad}^+(R) \subseteq \mathrm{Quad}(R)$ *be the full monoidal subcategory spanned by non-singular submodules of Euclidean spaces. Then*

$$K\left(\mathrm{Quad}^+(R)\right) \simeq K_0\left(\mathrm{Quad}^+(R)\right) \times BO_\infty(R)^+.$$

Note that if R is a field, or more generally a local ring, a non-singular quadratic module (M,q) lies in $\mathrm{Quad}^+(R)$ if and only if $q(x)$ is a sum of squares for each $x \in M$.

Appendix A. Arithmetic invariants of local rings

A.1. **Basic facts and definitions.** The unnamed invariant m_A is related to other ring invariants on which there is a rich and well-developed theory (see Lam's textbook [Lam05] for an overview). For each $n \in \mathbb{Z}_{\geq 1}$, let

$$S_n(A) = \{x_1^2 + \cdots + x_n^2 \mid x_1, \dots, x_n \in A\} \subseteq A.$$

Note that $S_n(A)$ is exactly the set of elements of A that are represented by Euclidean n-space $\mathbb{E}^n_A$. We define the invariants

$$\begin{aligned} &(\textit{Pythagoras number}) && P(A) = \inf\{p \in \mathbb{Z}_{\geq 1} \mid \bigcup_{n=1}^{\infty} S_n(A) = S_p(A)\}, \\ &(\textit{Stufe}) && s(A) = \inf\{s \in \mathbb{Z}_{\geq 1} \mid -1 \in S_s(A)\}, \\ & && u(A) = \sup\{\dim_{\mathbf{k}} \overline{V} \mid (V,q) \in \mathrm{Quad}(A) \text{ with no isotropic vector}\}. \end{aligned}$$

The invariant $u(A)$ is sometimes called the *u-invariant* of A.

The letter u comes from an alternative characterization, which we give now.

Definition. A non-singular quadratic module (V, q) over a commutative ring R is said to be *universal* if it represents every unit of R.

Proposition A.1. *Let A be a local ring with residue field $\mathbf{k}$, and assume that $2 \in A^\times$. Then $u(A)$ equals the smallest number u such that every non-singular quadratic module (V, q) with $\dim_{\mathbf{k}} \overline{V} \geq u$ is universal.*

Proof. Let (V, q) be a non-singular quadratic module with $\dim_{\mathbf{k}} \overline{V} \geq u(A)$. We claim that (V, q) is universal. Indeed, let $a \in A^\times$. Then $\dim_{\mathbf{k}} \overline{V} \oplus \mathbf{k} > u(A)$, so the non-singular quadratic module $(V \oplus A, q \mathbin{\textcircled{\perp}} \langle -a \rangle)$ is isotropic. By the representation theorem, we thus have that q represents a.

For the reverse inequality, suppose (V, q) is a non-singular quadratic module over A such that every non-singular quadratic module of dimension strictly less than $\dim_{\mathbf{k}} \overline{V}$ is universal. By the diagonalization theorem, we may assume that $q \cong \langle a_1, \ldots, a_n \rangle$ with $a_1, \ldots, a_n \in A^\times$. Then $\langle a_1, \ldots, a_{n-1} \rangle$ is universal, hence represents $-a_n$, and the representation theorem gives that $\langle a_1, \ldots, a_n \rangle$ is isotropic. □

It follows immediately from the proposition that

$$m_A \leq u(A) \quad \text{and} \quad s(A) \leq u(A).$$

Also, if $-1 = \sum_{j=1}^{s} a_j^2$, then for *any* element $x \in A$ we find

$$x = \left(\frac{x+1}{2}\right)^2 + \sum_{j=1}^{s} \left(\frac{a_j(x-1)}{2}\right)^2,$$

so x is a sum of $s + 1$ squares. In particular, this implies that

$$P(A) \leq s(A) + 1. \tag{A.1}$$

Note also that if $s(A) < \infty$, then $s(A) \leq P(A)$.

Proposition A.2.

(i) If F is a field, then F is Pythagorean if and only if $m_F = 1$.
(ii) $s(\mathbb{F}_q) \in \{1, 2\}$, $m_{\mathbb{F}_q} = P(\mathbb{F}_q) = u(\mathbb{F}_q) = 2$ for $q = p^n$, $p \neq 2$.
(iii) If F is a local or global field, then $m_F \leq 4$.

Proof. (i) Recall that a Pythagorean field is a field whose Pythagoras number is one. Suppose F is Pythagorean and let V be a non-singular nonzero subspace of $\mathbb{E}^n_F$. Pick an orthogonal basis $\{v_1, \ldots, v_k\}$ of V, and let $a_j = q(v_j)$ for each j. Each a_j is a sum of squares, so the assumption provides $b \in F$ with $b^2 = a_1$. Then $1/b \cdot v_1$ is a unit vector in V. Conversely, suppose $m_F = 1$ and let $a \in F$ be a sum of squares, say, $a = x_1^2 + \cdots + x_n^2$. We may assume $a \neq 0$ since $0 = 0^2$. Then the span of $v = (x_1, \ldots, x_n)$ is a non-singular subspace of $\mathbb{E}^n_F$, so by assumption there is $b \in F$ with $1 = n\langle 1 \rangle(bv) = b^2 a$, implying that $a = (1/b)^2$.

(ii) For the fact that every non-singular binary quadratic form over $\mathbb{F}_q$ is universal, see for instance [Lam05, Prop II.3.4]. To show the remaining statements, it suffices by (i) and the inequalities preceding the proposition to see that $P(\mathbb{F}_q) > 1$. Non-squares exist in $\mathbb{F}_q$, as otherwise the function $\mathbb{F}_q \to \mathbb{F}_q$, $x \mapsto x^2$, would be surjective and thus also injective.* But $1 = 1^2 = (-1)^2$

*I thank Kasper Andersen for teaching me this elementary argument.

and $1 \neq -1$ since q is not a power of two. On the other hand, since $u(\mathbb{F}_q) < \infty$, every element of $\mathbb{F}_q$ is a sum of squares; hence $P(\mathbb{F}_q) > 1$.

(iii) In the local field case, we invoke the classification of local fields. If F is $\mathbb{R}$ or $\mathbb{C}$, then F is pythagorean and we have $m_F = 1$ by (i). Otherwise F is a local field with finite residue class field and $u(F) \leq 4$ by [Lam05, Thm VI.2.12], so $m_F \leq u(F) \leq 4$. The global field case follows from the local field case by the Hasse–Minkowski theorem. □

Various other bounds on $P(F)$, $s(F)$ and $u(F)$ can be found in the literature on these invariants. We highlight the following difficult result, a proof of which is found in Scharlau's textbook [Sch12]:

Theorem A.3 (Tsen–Lang)**.** *Let F be a field of transcendence degree d over an algebraically closed field K. Then $u(F) \leq 2^d$.*

A.2. **Invariants of henselian local rings.** Knowing that there are plenty of fields with finite arithmetic invariants, we next consider how arithmetic properties of a local ring A relate to those of its residue field $\mathbf{k}$.

Proposition A.4. *Let A be a local ring with residue field $\mathbf{k}$, and assume that $2 \in A^\times$. Then*

$$m_{\mathbf{k}} \leq m_A, \tag{A.2}$$

$$P(\mathbf{k}) \leq P(A), \tag{A.3}$$

$$s(\mathbf{k}) \leq s(A), \tag{A.4}$$

$$u(\mathbf{k}) \leq u(A) \tag{A.5}$$

with equality in (A.2)*,* (A.4)*, and* (A.5) *if A is henselian.*

The proof will use two lemmata. The first is essentially [Bae78, Lem V.1.4]:

Lemma A.5. *Let A be a henselian local ring with $2 \in A^\times$ and let (V, q) be a non-singular quadratic module over A.*

(i) If the reduction $(\overline{V}, \overline{q})$ is isotropic, then (V, q) is isotropic.
(ii) Let $a \in A^\times$. If $(\overline{V}, \overline{q})$ represents $\overline{a}$, then (V, q) represents a.

Proof. (i) Since $\overline{V}$ is isotropic and non-singular, we may pick a hyperbolic pair $\overline{u}, \overline{v} \in \overline{V}$ with representatives $u, v \in V$. Put $a = q(u)$, $b = B_q(u, v)$ and $c = q(v)$ and consider the polynomial $f(X) = aX^2 + bX + c$. After reduction modulo $\mathfrak{m}$, we have $\bar{f}(X) = X$, so $\bar{f}(0) = 0$ and $\bar{f}'(X) = 1 \neq 0$. Using that A is henselian, we find a root $\lambda \in A$ for the unreduced polynomial. But then

$$q(\lambda u + v) = \lambda^2 q(u) + q(v) + \lambda B_q(u, v) = f(\lambda) = 0.$$

(ii) Follows from (i) and the representation theorem. □

Lemma A.6. *Let A be a local ring with residue field $\mathbf{k}$, and assume that $2 \in A^\times$. Let (V, q) be a quadratic module over A and let $\mathbf{U} \subseteq \overline{V}$ be a non-singular subspace of the reduction. Then there is a quadratic subspace $U \subseteq V$ whose reduction $\overline{U}$ is isometric to $\mathbf{U}$.*

Proof. Choose an orthogonal basis $\bar{u}_1, \dots, \bar{u}_n$ of $\mathbf{U} \subseteq \overline{V} = V/\mathfrak{m}V$ and pick representatives $u_1, \dots, u_n \in V$. Let $U = Au_1 + \cdots + Au_n \subseteq V$. We claim that U is a free A-module and that $u_1, \dots, u_n$ is a basis. Indeed, suppose $\sum_{i=1}^n a_i u_i = 0$ with $a_1, \dots, a_n \in A$. Applying $B_q(-, u_j)$ for $j = 1, \dots, n$, we get a system of n equations

$$\sum_{i=1}^n a_i B_q(u_i, u_j) = 0, \qquad j = 1, \dots, n.$$

Note that $\overline{\det(B_q(u_i, u_j))_{i,j}} = \det(B_{\overline{q}}(\bar{u}_i, \bar{u}_j)) \neq 0$ by non-singularity of $\mathbf{U}$, so $\det(B_q(u_i, u_j))_{i,j} \in A^\times$ and hence we must have $a_i = 0$ for each i by [Lan02, Prop XIII.4.16]. Thus U is free and the isometry $\overline{U} \cong \mathbf{U}$ is obvious. It follows from Propositions 1.1 and 1.4 that U is a submodule of V. □

Proof of Proposition A.4. Inequalities (A.3) and (A.4) follow from the easy observation that if $f\colon R \to S$ is a surjection of rings, then $P(S) \le P(R)$ and $s(S) \le s(R)$. To see (A.2), suppose $m \ge m_A$. If $\mathbf{V}$ is an m-dimensional non-singular subspace of $\mathbb{E}^n_{\mathbf{k}}$ for some n, then Lemma A.6 says there is a non-singular subspace $V \subseteq \mathbb{E}^n_A$ with $\overline{V} \cong \mathbf{V}$. But then by assumption V contains a unit vector, and hence so does $\mathbf{V}$. To see (A.5), let $(\mathbf{V}, q)$ be an arbitrary d-dimensional non-singular quadratic module over $\mathbf{k}$ with $d \ge u(A)$. Pick an orthogonal basis $\mathbf{V} \cong \langle \bar{a}_1, \dots, \bar{a}_d\rangle$. Then $V = \langle a_1, \dots, a_d\rangle$ is a d-dimensional non-singular quadratic module over A, hence is universal. It follows that $\mathbf{V} \cong \overline{V}$ is universal.

Suppose now that A is henselian. If V is a non-singular quadratic subspace of $\mathbb{E}^n_A$ for some n having $\dim_{\mathbf{k}} \overline{V} \ge m_{\mathbf{k}}$, then the reduction $\overline{V}$ is a non-singular quadratic subspace of $\mathbb{E}^n_{\mathbf{k}}$. Since $\dim_{\mathbf{k}} \overline{V} \ge m_{\mathbf{k}}$, we have that $\overline{V}$ contains a unit vector, and Corollary A.5 implies that V contains a unit vector. Similar applications of Corollary A.5 show that equalities hold in (A.4) and (A.5). □

Remark. (i) It is far from true that equality holds in (A.3) if A is henselian. Indeed, one very quickly runs into $P(A) = \infty$ if finiteness of $P(A)$ is not forced by finiteness of the Stufe. Recall that a field is called *formally real* if its Stufe is infinite. Choi, Dai, Lam and Reznick [CDLR82] have shown that if R is a regular local ring with $\dim_{\mathrm{Krull}}(R) \ge 3$ whose residue field is formally real, then $P(R) = \infty$. In particular, if $\mathbf{k}$ is formally real then the complete local ring $A = \mathbf{k}[\![X_1, \dots, X_d]\!]$ has $P(A) = \infty$ for all $d \ge 3$.
(ii) Some condition on the local ring A is necessary for equality to hold in (A.2), (A.4), and (A.5). Let p be an odd prime. The ring $\mathbb{Z}_{(p)}$ has $s(\mathbb{Z}_{(p)}) = u(\mathbb{Z}_{(p)}) = \infty$, but $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{F}_p$ has $s(\mathbb{F}_p) \le 2$ and $u(\mathbb{F}_p) = 2$, which shows that a condition is necessary for equality to hold in (A.4) and (A.5). Furthermore, $m_{\mathbb{F}_p} = 2$ whereas $m_{\mathbb{Z}_{(p)}} = 4$, showing that we cannot in general expect equality to hold in (A.2). To see that $m_{\mathbb{Z}_{(p)}} \ge 4$, pick $a, b \in \mathbb{Z}_{\ge 1}$ such that $p \nmid 4^a(8b-1)$; for instance, one could take $a = 0$ and $b = p$. Then $3\left\langle \frac{1}{4^a(8b-1)}\right\rangle$ is a non-singular subspace of $\mathbb{E}^{12}_{\mathbb{Z}_{(p)}}$ since $\frac{1}{4^a(8b-1)}$ is a sum of four squares in $\mathbb{Z}_{(p)}$ by Lagrange's four-square theorem. Indeed, the four-square theorem says there is $r_1, r_2, r_3, r_4 \in \mathbb{Z}$ with $r_1^2 + r_2^2 + r_3^2 + r_4^2 = 4^a(8b-1)$. But then

$$\left(\frac{r_1}{4^a(8b-1)}\right)^2 + \left(\frac{r_2}{4^a(8b-1)}\right)^2 + \left(\frac{r_3}{4^a(8b-1)}\right)^2 + \left(\frac{r_4}{4^a(8b-1)}\right)^2 = \frac{1}{4^a(8b-1)},$$

and $\frac{r_j}{4^a(8b-1)} \in \mathbb{Z}_{(p)}$ for each j. However, $3\left\langle \frac{1}{4^a(8b-1)}\right\rangle$ does not contain a unit vector. To see this, assume for contradiction that $(x, y, z) \in \mathbb{Z}^3_{(p)} \subseteq \mathbb{Q}^3$ is a unit vector, i.e. that

$$\frac{x^2 + y^2 + z^2}{4^a(8b-1)} = 1.$$

Clearing the denominator, we find

$$x^2 + y^2 + z^2 = 4^a(8b-1).$$

But (the trivial direction of) Legendre's three-square theorem together with the Davenport–Cassels theorem say that $4^a(8b-1)$ cannot be expressed as a sum of three squares of rational numbers, contradiction. In fact it follows from Proposition A.8 below that $m_{\mathbb{Z}_{(p)}} \le m_{\mathbb{Q}} = 4$, and hence $m_{\mathbb{Z}_{(p)}} = 4$ as claimed.

A.3. **Invariants of valuation rings.** Finally, we consider how the arithmetic invariants of a valuation ring are also related to those of its quotient field.

Lemma A.7. *Let A be a valuation ring with $2 \in A^\times$, and let (V, q) be a non-singular quadratic module over A. Let K be the quotient field of A and denote by (V_K, q_K) the quadratic space over K with $V_K = K \otimes_A V$ and $q_K(\alpha \otimes x) = \alpha^2 q(x)$ for $\alpha \in K$, $x \in V$. Then*

(i) (V, q) is isotropic if and only if (V_K, q_K) is.
(ii) Then (V, q) represents $a \in A^\times$ if and only if (V_K, q_K) does.

Proof. Suppose (V_K, q_K) is isotropic. We may assume that $V = A^n$. Let $0 \neq x \in V$ with $q_K(x) = 0$, and let $\nu\colon K \to \Gamma \cup \{\infty\}$ be a valuation for A. Write $x = (x_1, \ldots, x_n)$ and pick x_i with $\nu(x_i) \leq \nu(x_j)$ for all j. Then $x_i^{-1}x \in A^n$ since $\nu(x_i^{-1}x_j) = \nu(x_j) - \nu(x_i) \geq 0$ for each j. Also, $x_i^{-1}x$ is primitive since its ith coordinate is $x_i^{-1}x_i = 1$. But

$$q(x_i^{-1}x) = q_K(x_i^{-1}x) = x_i^{-2}q_K(x) = 0.$$

This proves the nontrivial direction of (i). Further, (ii) follows from (i) via the representation theorem. □

Proposition A.8. *Let A be a valuation ring with $2 \in A^\times$, and denote the quotient field of A by K. Then*

$$m_A \leq m_K, \tag{A.6}$$

$$s(A) = s(K), \tag{A.7}$$

and

$$u(A) \leq u(K) \tag{A.8}$$

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