THE DERIVED CATEGORY OF A LOCALLY COMPACT SPACE IS RARELY SMOOTH

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ABSTRACT. We show that the derived category of a locally compact Hausdorff space X (subject to a mild technical condition) is smooth in the sense of non-commutative geometry if and only if X is discrete and finite.

In this note, we consider the derived category of sheaves on a locally compact Hausdorff space from the point of view of *non-commutative geometry* à la Kontsevich [Kon05]. The motivation for non-commutative geometry comes from algebraic geometry, and we start by briefly recalling some facts from that setting. In order to reconstruct a scheme X from its associated dg-category of perfect complexes $\mathcal{D}_{\mathrm{perf}}(X)$, it is generally necessary to remember also the tensor structure on the latter.[1] Nevertheless, Kontsevich has shown that some geometric properties of X are remembered by the dg-category $\mathcal{D}_{\mathrm{perf}}(X)$ viewed without its monoidal structure, or in other words as an object of the ∞-category of *non-commutative spaces*, which is a fancy name for the (very large) ∞-category of stable presentable ∞-categories and colimit-preserving functors $\mathcal{P}\mathrm{r}_{\mathrm{st}}$. In particular, under mild conditions on X one has that the stable ∞-category $\mathcal{D}_{\mathrm{perf}}(X)$ detects whether X is smooth (resp. proper) [Kon05], see also [Lur18, Ch 11]. Thus the notions of smoothness and properness belonging to algebraic geometry extend to the setting of non-commutative spaces. Recall that a colimit-preserving functor $f^*\colon \mathcal{C} \to \mathcal{D}$ between stable presentable ∞-categories is *strongly continuous* if its right adjoint f_* admits a further right adjoint $f^!$.[2] We then have the following definition of Kontsevich [Kon05] (cf. [Lur18, Ste23]):

Definition 0.1. Let $\mathcal{C} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}_{\mathrm{st}}^{\otimes})$ be a presentably symmetric monoidal stable ∞-category. A *smooth (resp. proper) $\mathcal{C}$-linear category* is a presentable stable $\mathcal{C}$-linear ∞-category $\mathcal{M} \in \mathrm{Mod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}_{\mathrm{st}}^{\otimes})$ such that

(i) $\mathcal{C}$ is dualizable with respect to the Lurie tensor product $\otimes_{\mathcal{C}}$ on $\mathrm{Mod}_{\mathcal{C}}(\mathcal{P}\mathrm{r}_{\mathrm{st}})$; and
(ii) The coevaluation functor $\mathcal{C} \to \mathcal{M} \otimes_{\mathcal{C}} \mathcal{M}^{\vee}$ (resp. the evaluation functor $\mathcal{M}^{\vee} \otimes_{\mathcal{C}} \mathcal{M} \to \mathcal{C}$) is strongly continuous.

If R is an $\mathbb{E}_{\infty}$-ring, we will abbreviate 'Mod_R-linear' to 'R-linear'.

Non-commutative spaces from point-set topology. Non-commutative spaces form a world in which objects from algebraic geometry live alongside objects from point-set topology. Namely, for each topological space X and each stable presentable ∞-category $\mathcal{C}$, the ∞-category $\mathrm{Shv}(X, \mathcal{C})$ of $\mathcal{C}$-valued sheaves on X (in the sense of Lurie [Lur09]) will again be a stable presentable ∞-category. In this setting, Aoki has recently announced an analog of the reconstruction theorem for schemes that we indicated above, according to which a locale X is canonically isomorphic to the locale of localizing $\otimes$-ideals in the symmetric monoidal ∞-category $(\mathrm{Shv}(X, \mathcal{D}(k)), \otimes)$, where $\mathcal{D}(k)$ is the unbounded derived ∞-category of a field k [Aok23]. In particular, this gives a

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[1]See [Muk81, Orl97, Pol96] for examples showing that X cannot be reconstructed from $\mathcal{D}_{\mathrm{perf}}(X)$ without the monoidal structure. The fact that any topologically Noetherian reduced scheme X *can* be reconstructed from $(\mathcal{D}_{\mathrm{perf}}(X), \otimes)$ is due to Balmer [Bal02], building on earier results by Hopkins–Neeman–Thomason.

[2]In the unstable setting, one must only require that the right adjoint f_* preserves filtered colimits. If both the source and target of f_* are stable, this is equivalent to f_* preserving all (small) colimits, or equivalently admitting a right adjoint $f^!$.

way to reconstruct any sober space X from the derived category $\mathrm{Shv}(X, \mathcal{D}(k))$ and its associated tensor structure.

This paper is concerned with the following question:

Question 1. How useful are the tools from non-commutative geometry à la Kontsevich for studying non-commutative spaces coming from point-set topology as above?

We will focus on the following aspect of the previous question:

Question 2. How useful is the notion of *smoothness* (in the sense of Definition 0.1) for studying topological spaces?

The answer to this question turns out to be:

$$\textit{not useful at all!}$$

Indeed, we show:

Theorem 0.2. *Let X be a* Sp-*hypercomplete[3] locally compact Hausdorff space. If* $\mathrm{Shv}(X, \mathrm{Sp})$ *is a smooth* $\mathbb{S}$-*linear category, then X is finite.*

In particular, any locally compact space which is smooth in the sense of non-commutative geometry must be discrete. As a corollary, we recover the following result of Ramzi [Ram22]:

Corollary 0.3 (Ramzi)**.** *Let M be a topological manifold such that* $\mathrm{Shv}(M, \mathrm{Sp})$ *is a smooth* $\mathbb{S}$-*linear category. Then M is discrete.*

In fact, the proof of Theorem 0.2 boils down to combining the ideas in [Ram22] with some recent work by the author [Har23].

Remark 0.4. Using the results from [Har23], we can also give a proof of Corollary 0.3 which is independent of the ideas in [Ram22]. We include this proof here, as it provides some intuition for why smoothness should imply discreteness.

The coevaluation for $\mathrm{Shv}(M, \mathrm{Sp})$ is (equivalent to)

$$\mathrm{Sp} \xrightarrow{\Gamma^!} \mathrm{Shv}(M, \mathrm{Sp}) \xrightarrow{\Delta_*} \mathrm{Shv}(M \times M, \mathrm{Sp}) \simeq \mathrm{Shv}(M, \mathrm{Sp}) \otimes \mathrm{Shv}(M, \mathrm{Sp})^\vee,$$

where $\Gamma \colon M \to *$ is the projection to a point and $\Delta \colon M \to M \times M$ is the diagonal immersion. If $\Delta_* \Gamma^!$ is strongly continuous, it must preserve compact objects, so in particular $\Delta_* \Gamma^! \mathbb{S}$ must be compact, where $\mathbb{S}$ is the sphere spectrum. (In fact this is equivalent to the coevaluation being strongly continuous by [Lur09, Prop 5.5.7.2].) By [Har23, Thm 2.3], this is equivalent to $\Delta_* \Gamma^! \mathbb{S}$ (i) having compact support, (ii) being locally constant, and (iii) having compact stalks. In particular, (i) and (ii) together imply that the support $\mathrm{supp}\, \Delta_* \Gamma^! \mathbb{S}$ must be compact open as a subset of $M \times M$. Let $U \subseteq M$ be an open subset such that $U \cong \mathbb{R}^d$, and let $j \colon U \hookrightarrow M$ denote the inclusion. By [Vol23, Prop 6.18], we have $j^! \Gamma^! \mathbb{S} \simeq \Gamma_U^! \mathbb{S} \simeq \Sigma^d \mathbb{S}$, where $\Gamma_U \colon U \to *$ is the projection to a point.[4] We thus find

$$(\Delta_* \Gamma^! \mathbb{S})_{(x,y)} \simeq \begin{cases} \Sigma^d \mathbb{S}, & \text{if } x = y, \\ 0, & \text{else.} \end{cases}$$

Since $\Sigma^d \mathbb{S} \not\simeq 0$, we then have that $\mathrm{supp}\, \Delta_* \Gamma^! \mathbb{S} = \Delta(M) \subseteq M \times M$. Since $\Delta(M)$ is an open subset of $M \times M$, then M must be discrete. Since $\Delta(M)$ is also compact, then M must be finite, completing the proof.

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[3]As in [Har23], we say that a topological space X is $\mathcal{C}$-*hypercomplete*, where $\mathcal{C}$ is some ∞-category, if equivalences of $\mathcal{C}$-valued sheaves on X are detected stalkwise.

[4]In fact, if M is a smooth manifold then $\Gamma^! \mathbb{S}$ is exactly the 'dualizing sheaf' that underlies Atiyah duality, as explained by the following theorem of Volpe: there is a canonical equivalence $\Gamma^! \mathbb{S} \simeq \mathrm{Th}(\tau_M)$, where $\mathrm{Th}(\tau_M)$ is the M-parametrized Thom spectrum of the tangent bundle $\tau_M \colon TM \to M$ [Vol23, Thm 7.11].

Fully dualizable categories. We will briefly explain the original motivation for this note. Let R be an $\mathbb{E}_\infty$-ring, and write $\mathcal{P}r_{R,\mathrm{st}} = \mathrm{Mod}_{\mathrm{Mod}_R}(\mathcal{P}r_{\mathrm{st}}^\otimes)$ for the ∞-category of presentable R-linear ∞-categories. Note that a dualizable R-linear category $\mathcal{M}$ is smooth and proper if and only if it is dualizable–not just as an object of $\mathcal{P}r_{R,\mathrm{st}}$–but also as an object of the non-full monoidal subcategory $\mathcal{P}r_{R,\mathrm{st}}^{\mathrm{d}}$ consisting of dualizable R-linear ∞-categories and strongly continuous functors. If $\mathcal{M}$ satisfies this condition, it is said to be a *fully dualizable R-linear category*. With the additional assumption of being compactly generated, the study of fully dualizable categories was initiated by Kontsevich [Kon05] (under the name of 'saturated non-commutative spaces'), and remains an active area of research [Orl18, Orl20]. Less mysterious now is the class of *invertible* compactly generated R-linear ∞-categories, which have been classified by Antieau–Gepner [AG14], building on earlier work of Toën in the setting of simplicial rings [Toë12].

Recent developments suggest that when possible one should work with all dualizable stable ∞-categories, and not just those that are compactly generated. For instance:

(1) Efimov has shown that the usual K-theory functor

$$K \colon \mathcal{P}r_{\mathrm{st}}^{\mathrm{cg}} \to \mathrm{Sp},$$

previously defined on the ∞-category of compactly generated presentable stable ∞-categories and strongly continuous functors between them (see e.g. [BGT13]), admits an extension

$$
\begin{array}{ccc}
\mathcal{P}r_{\mathrm{st}}^{\mathrm{cg}} & \longhookrightarrow & \mathcal{P}r_{\mathrm{st}}^{\mathrm{d}} \\
{\scriptstyle K}\downarrow & \swarrow{\scriptstyle K_{\mathrm{cont}}} & \\
\mathrm{Sp} & &
\end{array}
$$

to the larger ∞-category $\mathcal{P}r_{\mathrm{st}}^{\mathrm{d}} \supset \mathcal{P}r_{\mathrm{st}}^{\mathrm{cg}}$, recovering interesting invariants for non-compactly generated ∞-categories coming from topology, geometry, functional analysis, etc. [Efi22, Hoy18].

(2) Ramzi has shown that the ∞-category of dualizable categories $\mathcal{P}r_{\mathrm{st}}^{\mathrm{d}}$ has very good categorical properties, which are not enjoyed by the subcategory $\mathcal{P}r_{\mathrm{st}}^{\mathrm{cg}}$ of compactly generated stable ∞-categories. In particular, the ∞-category $\mathcal{P}r_{\mathrm{st}}^{\mathrm{d}}$ is presentable [Ram].

It is therefore natural to expand the programs of Kontsevich–Orlov and Antieau–Gepner–Toën to include non-compactly generated dualizable categories. In this direction, Stefanich has classified the invertible objects of $\mathcal{P}r_{R,\mathrm{st}}^{\mathrm{d}}$ when R is an Artinian $\mathbb{E}_\infty$-ring [Ste23, Thm 1.0.7]; it is a consequence of Stefanich's theorem that these are all compactly generated, and hence agree with the invertible categories classified by Antieau–Gepner. Conjecturally, this could be because full dualizability is such a severe condition that it forces compact generation:

Conjecture 0.5. *If $\mathcal{C} \in \mathcal{P}r_{R,\mathrm{st}}^{\mathrm{d}}$ is fully dualizable, then it is compactly generated.*

One purpose of this note is to test this conjecture against a well-known class of dualizable ∞-categories that are rarely compactly generated, namely categories of derived sheaves on locally compact Hausdorff spaces [Nee01, Har23]. Modulo a hypercompleteness assumption, we verify the conjecture for this class of dualizable ∞-categories (Lemma 1.1).

1. Smooth spaces are totally disconnected

Fix a nontrivial compactly generated stable ∞-category $\mathcal{C} \not\simeq 0$, e.g. the unbounded derived ∞-category of a ring $\mathcal{D}(R)$ or the ∞-category of spectra.

In order to prove Theorem 0.2, we will first prove the following weaker statement:

Lemma 1.1. *Let X be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space. If $\mathrm{Shv}(X,\mathcal{C})$ is smooth, then X is totally disconnected.*[5]

The idea of the proof is to use an observation of Ramzi [Ram22], according to which X is in a sense 'generated' by a finite cover, and then proceed by induction on the Krull dimension of the nerve poset of the cover to reduce to the case in which the cover only has one member.

Remark 1.2. This is the part of the argument that is relevant for Conjecture 0.5, as $\mathrm{Shv}(X,\mathcal{C})$ is compactly generated whenever X is totally disconnected by a result of Lurie [Lur09, Prop 6.5.4.4]; in fact, it was shown in [Har23] that $\mathrm{Shv}(X,\mathcal{C})$ is compactly generated if *and only if X is totally disconnected*, as long as X is $\mathcal{C}$-hypercomplete.

1.1. **A finite 'generating' cover.** Given a topological space X, an open subspace $U \subseteq X$, and an object $C \in \mathcal{C}$, we let $C[U]$ denote the sheafification of the $\mathcal{C}$-valued presheaf defined by

$$V \mapsto \begin{cases} C, & \text{if } V \subseteq U, \\ 0, & \text{else,} \end{cases}$$

for $V \subseteq X$ open. Alternatively, if $j \colon U \hookrightarrow X$ denotes the inclusion, then $C[U] = j_! C_U$, where C_U is the constant sheaf on U with value C.

Lemma 1.3. *The ∞-category $\mathrm{Shv}(X,\mathcal{C})$ is generated under colimits by the sheaves $C[U]$, as U varies over open subsets of X and C varies over compact objects of $\mathcal{C}$.*

Proof. Given an open subspace $U \subseteq X$, we let $\chi_U \in \mathrm{Shv}(X)$ denote the sheafification of the presheaf of anima given by

$$V \mapsto \begin{cases} \Delta^0, & \text{if } V \subseteq U, \\ 0, & \text{else,} \end{cases}$$

for $V \subseteq X$ open. The sheaves χ_U generate $\mathrm{Shv}(X)$ under colimits, e.g. since this is true at the level of presheaves by [Lur09, Cor 5.1.5.8]. One checks that there is a functor $\mathrm{Shv}(X) \times \mathcal{C} \to \mathrm{Shv}(X,\mathcal{C})$ uniquely determined by the facts that it preserves colimits in each variable and sends $\chi_U \times C$ to $C[U]$ for $C \in \mathcal{C}$. By the universal property of the Lurie tensor product, this functor factors as

$$\mathrm{Shv}(X) \times \mathcal{C} \xrightarrow{p} \mathrm{Shv}(X) \otimes \mathcal{C} \xrightarrow{q} \mathrm{Shv}(X,\mathcal{C}),$$

where p is the canonical functor and q is the equivalence of [Lur18, Rmk 1.3.1.6]. Recall that the Lurie tensor product is 'spanned by simple tensors' (see e.g. [Ram22, Lem 0.1]), so $\mathrm{Shv}(X) \otimes \mathcal{C}$ is generated under colimits by the image of the functor p. Since $\mathcal{C}$ is generated under colimits by the compact objects $\mathcal{C}^\omega \subseteq \mathcal{C}$ and $\mathrm{Shv}(X)$ is generated under colimits by the sheaves χ_U, we find that $\mathrm{Shv}(X) \otimes \mathcal{C}$ is generated under colimits by objects of the form $p(\chi_U \times C)$ for $U \subseteq X$ open and $C \in \mathcal{C}^\omega$. Hence $\mathrm{Shv}(X,\mathcal{C})$ is generated by sheaves of the form $qp(\chi_U \times C) \simeq C[U]$ as desired. $\qquad\square$

The following result is due to Ramzi [Ram22]:

Proposition 1.4 (Ramzi). *Let $\mathcal{D}$ be a smooth presentable stable ∞-category and let S be a set of objects in $\mathcal{C}$. If S generates $\mathcal{D}$ under colimits, then there is a finite subset $S' \subseteq S$ such that S' generates $\mathcal{D}$ as a localizing subcategory of itself.*[6]

Corollary 1.5. *Let X be a topological space such that $\mathrm{Shv}(X,\mathcal{C})$ is smooth. Then there is a finite collection of open sets $U_1, U_2, \ldots, U_n \subseteq X$ so that the sheaves $C[U_i]$ generate $\mathrm{Shv}(X,\mathcal{C})$ generate $\mathrm{Shv}(X,\mathcal{C})$ as a localizing subcategory of itself, where i varies over $\{1, \ldots, n\}$ and C varies over compact objects of $\mathcal{C}$.*

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[5]Recall that a topological space X is *totally disconnected* if and only if for each pair of distinct points $x, y \in X$, there is a clopen neighborhood $U \ni x$ with $y \notin U$. If X is in addition locally compact, then an elementary argument using precompact neighborhoods shows that X is totally disconnected if and only if for each point $x \in X$ and each open neighborhood $U \ni x$, there is a smaller neighborhood V, $x \in V \subseteq U$, such that V is clopen.

[6]That is, if $\mathcal{D}' \subseteq \mathcal{D}$ is a localizing subcategory containing S', then $\mathcal{D}' = \mathcal{D}$. Recall that a full subcategory $\mathcal{D}' \subseteq \mathcal{D}$ is *localizing* if it is closed under (small) colimits and finite limits, or in other worlds if it is both a stable subcategory and closed under colimits.

Proof. This follows immediately from the previous proposition and Lemma 1.3. $\square$

Remark 1.6. Let X and $U_1, U_2, \ldots, U_n \subseteq X$ be as in the proceeding corollary. Then $X = \bigcup_1^n U_i$. Indeed, if $x \in X$, then the full subcategory $\{\mathscr{F} \mid \mathscr{F}_x \simeq 0\} \subseteq \mathrm{Shv}(X, \mathcal{C})$ is localizing, since the stalk functor is left exact and preserves (small) colimits. Note also that $\{\mathscr{F} \mid \mathscr{F}_x \simeq 0\} \neq \mathrm{Shv}(X, \mathcal{C})$ since the skyscraper sheaf $x_* D$ at any nonzero object $D \in \mathcal{C}$ has $(x_* D)_x \simeq D \not\simeq 0$. Now observe that if $x \notin U_i$, then $C[U_i] \in \{\mathscr{F} \mid \mathscr{F}_x \simeq 0\}$ for each C, and hence there must be some j with $x \in U_j$.

1.2. **Digression: The Makkai–Pitts criterion.** There is a well-known useful criterion for checking whether a collection of compact objects generates an ∞-category, which in the setting of ordinary categories is due to Makkai and Pitts [MP87, Lem 1.7]. The statement carries over to higher categories without a hitch, but we have not been able to find an account of this piece of folklore in the literature.

Proposition 1.7 (Makkai–Pitts). *Let κ be a regular cardinal and let $\mathcal{D}$ be a locally small ∞-category that admits all κ-filtered colimits. Suppose that $\mathcal{D}^0 \subseteq \mathcal{D}$ is a small full subcategory consisting of κ-compact objects such that $\mathcal{D}^0$ admits all κ-small colimits, and furthermore that these colimits are preserved by the inclusion into $\mathcal{D}$.*
    If the functors

$$\mathrm{Map}_{\mathcal{D}}(C, -) \colon \mathcal{D} \to \mathcal{S}$$

are jointly conservative for C varying over $\mathcal{D}^0$, then $\mathcal{D}$ is generated under filtered colimits by $\mathcal{D}^0$. In particular, the ∞-category $\mathcal{D}$ is κ-compactly generated.

Proof. We use the yoga of Ind-categories, as developed in [Lur09, § 5.3.5]. Recall that if $\mathcal{E}$ is some small ∞-category, then $\mathrm{Ind}_\kappa(\mathcal{E})$ is the ∞-category defined by freely adjoining κ-filtered colimits to $\mathcal{E}$. Since $\mathcal{D}^0$ admits κ-small colimits, we may identify $\mathrm{Ind}_\kappa(\mathcal{D}^0)$ with the full subcategory of $\mathrm{PShv}(\mathcal{D}^0)$ spanned by presheaves of anima $F \colon (D^0)^{\mathrm{op}} \to \mathcal{S}$ that preserve κ-small limits [Lur09, Cor 5.3.5.4].
    Let $y \colon \mathcal{D} \hookrightarrow \mathrm{PShv}(\mathcal{D})$ denote the Yoneda embedding and let $i^* \colon \mathrm{PShv}(\mathcal{D}) \to \mathrm{PShv}(\mathcal{D}^0)$ denote restriction along the inclusion $i \colon \mathcal{D}^0 \hookrightarrow \mathcal{D}$. The fact that i preserves κ-small colimits means that $i^* y(D) = \mathrm{Map}_{\mathcal{D}}(-, D)|_{\mathcal{D}^0}$ belongs to the subcategory $\mathrm{Ind}_\kappa(\mathcal{D}^0) \subseteq \mathrm{PShv}(\mathcal{D}^0)$ for each $D \in \mathcal{D}$. Thus we have a functor $R \colon \mathcal{D} \to \mathrm{Ind}_\kappa(\mathcal{D}^0)$ given informally by

$$R \colon D \mapsto \mathrm{Map}_{\mathcal{D}}(-, D)|_{\mathcal{D}^0}.$$

Furthermore, since $\mathcal{D}$ admits κ-filtered colimits, the proof of [Lur09, Lem 5.3.5.8] shows that R admits a left adjoint $L \colon \mathrm{Ind}_\kappa(\mathcal{D}^0) \to \mathcal{D}$. The fact that the objects of $\mathcal{D}^0$ are κ-compact in $\mathcal{D}$ implies that L is fully faithful [Lur09, Prop 5.3.5.11]. Finally, the conservativity assumption precisely means that R is conservative (for this we also use the hard-won fact that equivalences in the presheaf ∞-category are detected pointwise, see e.g. [Lan21, Cor 2.2.2]), and the result now follows from the elementary observation below. $\square$

Lemma 1.8. *Let $L \colon \mathcal{D} \rightleftarrows \mathcal{E} \colon R$ be a pair of adjoint functors. If L is fully faithful and R is conservative, then L is an equivalence of categories.*

Proof. Given $E \in \mathcal{E}$, it will suffice to show that the unit $\eta_E \colon E \to LR(E)$ is an equivalence. By the triangle identity, the composition

$$R(E) \xrightarrow{R(\eta_E)} RLR(E) \xrightarrow{\epsilon_{R(E)}} R(E)$$

is homotopic to the identity $\mathrm{id}_{R(E)}$, where $\epsilon_{R(E)}$ is the counit. Since L is fully faithful we have that $\epsilon_{R(E)}$ is an equivalence, and then by the two-out-of-three property for equivalences we find that $R(\eta_E)$ must be an equivalence. Since R is conservative, it follows that η_E is an equivalence as desired. $\square$

Corollary 1.9. *Let $\mathcal{D}$ be a cocomplete stable ∞-category. If there is a small set S of compact objects that generate $\mathcal{D}$ as a localizing subcategory of itself, then $\mathcal{D}$ is compactly generated.*

Proof. Let $\mathcal{D}^0 \subseteq \mathcal{D}$ denote the smallest full subcategory of $\mathcal{D}$ containing S which is closed under finite limits and colimits. Then $\mathcal{D}^0$ again consists of compact objects by [Lur09, Cor 5.3.4.15].

We claim that the functors $\mathrm{Map}_{\mathcal{D}}(C, -) \colon \mathcal{D} \to \mathcal{S}$ are jointly conservative as C varies over $\mathcal{D}^0$. Since $\mathcal{D}$ is stable, it suffices to show that if $D \in \mathcal{D}$ has $\mathrm{Map}_{\mathcal{D}}(C, D)$ contractible for each $C \in S$, then $D \simeq 0$. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the full subcategory spanned by objects $E \in \mathcal{D}$ with $\mathrm{Map}_{\mathcal{D}}(E, D)$ contractible. By assumption $\mathcal{D}^0 \subseteq \mathcal{D}'$. Note also that $\mathcal{D}'$ is localizing, and so we must have $\mathcal{D}' = \mathcal{D}$. In particular, $\mathrm{Map}_{\mathcal{D}}(D, D)$ is contractible, so id_D is homotopic to 0, and hence $D \simeq 0$ as desired.

Finally, observe that $\mathcal{D}^0$ satisfies all the assumptions of Proposition 1.7, giving the desired conclusion. $\qquad\square$

1.3. **The inductive argument.** Above we saw that if the category $\mathrm{Shv}(X, \mathcal{C})$ is smooth, then it must be 'generated by a finite cover'. The argument now proceeds by an induction on the dimension of the Alexandrov nerve associated to this cover.

Recall that a poset P has a canonical topology called the *Alexandrov topology*, such that the sets $P_{\geq p} = \{q \in P \mid q \geq p\}$, $p \in P$, form a basis of open sets. A *P-stratification* of a topological space X is a continuous map $\phi \colon X \to P$. It is standard to denote

$$X_{\geq p} = \phi^{-1}(P_{\geq p}) \quad \text{and} \quad X_p = \phi^{-1}(\{p\}),$$

and to refer to the (locally closed) subsets $X_p \subseteq X$ as the *strata* associated to the stratification.

Construction 1.10 (Stratification over the Alexandrov nerve). Let X be a topological space and let $\{U_i\}_{i \in I}$ be a finite open cover of X. Consider the power set $\mathrm{Pow}(I)$ as a poset under inclusion. The *nerve poset of* $\{U_i\}_{i \in I}$ is the subposet

$$P = \{J \subseteq I \mid \cap_{j \in J} U_j \neq \varnothing\} \subseteq \mathrm{Pow}(I).$$

Define a function $X \to P$ by

$$x \mapsto \{i \in I \mid x \in U_i\} \subseteq I.$$

This function is continuous with respect to the Alexandrov topology on P, since the preimage of $P_{\geq J}$ is by construction $\cap_J U_j$. Thus it defines a P-stratification of X.
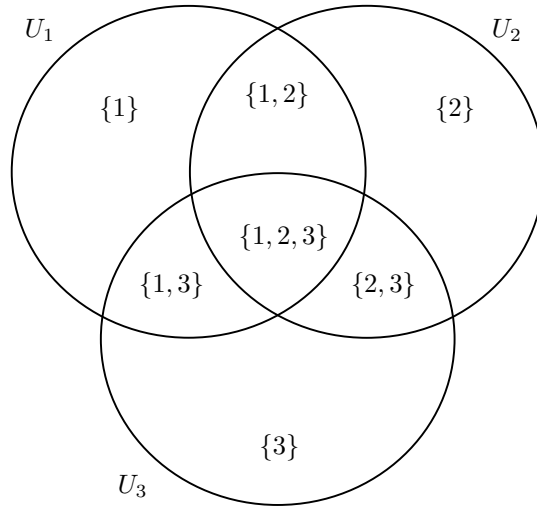

FIGURE 1. The stratification associated to a cover $X = U_1 \cup U_2 \cup U_3$.

In the inductive argument, the base case is essentially handled by the following lemma:

Lemma 1.11. *Let X be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space such that $\mathrm{Shv}(X, \mathcal{C})$ is generated as a localizing subcategory of itself by the sheaves $C[X]$ for $C \in \mathcal{C}$ compact. Then X is totally disconnected.*

Proof. Since X is covered by precompact open sets, it will suffice to show that any precompact open $U \subseteq X$ is totally disconnected. The restriction functor $\mathrm{Shv}(X, \mathcal{C}) \to \mathrm{Shv}(\overline{U}, \mathcal{C})$ is essentially surjective and preserves colimits, so $\mathrm{Shv}(\overline{U}, \mathcal{C})$ is generated as a localizing subcategory by sheaves of the form $C[\overline{U}]$ for $C \in \mathcal{C}^\omega$. But these sheaves are compact e.g. by [Har23, Thm 2.3], and hence $\mathrm{Shv}(\overline{U}, \mathcal{C})$ is compactly generated by Corollary 1.9. It follows that $\overline{U}$ is totally disconnected by [Har23, Prop 3.1], and hence so is $U \subseteq \overline{U}$ as desired. $\qquad\square$

Proposition 1.12. *Let X be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space. Suppose that there is a finite poset P and a stratification $X \to P$, such that $\mathrm{Shv}(X, \mathcal{C})$ is generated as a localizing subcategory of itself by sheaves of the form $C[X_{\geq p}]$, for $p \in P$ and $C \in \mathcal{C}$ compact. Then X is totally disconnected.*

Proof. The proof is by induction on the Krull dimension of P.[7] If $\dim P = 0$, then

$$X = \coprod_{p \in P} X_p = \coprod_{p \in P} X_{\geq p}.$$

Fix $p \in P$. Since the restriction functor $\mathrm{Shv}(X, \mathcal{C}) \to \mathrm{Shv}(X_{\geq p}, \mathcal{C})$ is essentially surjective and preserves (small) colimits, we find that $\mathrm{Shv}(X_{\geq p}, \mathcal{C})$ is generated as a localizing subcategory of itself by the sheaves $C[X_{\geq q}]|_{X_{\geq p}}$. Note however that $C[X_{\geq q}]|_{X_{\geq p}} \simeq 0$ if $p \neq q$, so in fact $\mathrm{Shv}(X_{\geq p}, \mathcal{C})$ must be generated as a localizing subcategory by the constant sheaves $C[X_{\geq p}]$ for C compact. Hence $X_{\geq p}$ is totally disconnected by Lemma 1.11, and hence X is also totally disconnected.

Now suppose $\dim P > 0$ and that the statement is known for stratifications over lower-dimensional posets. Let $P_0 = \max P \subset P$ denote the subset of maximal elements, and put

$$U = \coprod_{p \in P_0} X_p \quad \text{and} \quad Z = X \setminus U.$$

Then the stratified spaces $U \to P_0$ and $Z \to P \setminus P_0$ are covered by the induction hypothesis, and hence must be totally disconnected. The conclusion now follows from the fact that totally disconnected locally compact Hausdorff spaces are closed under recollement, as showed by the lemma below. $\qquad\square$

Lemma 1.13. *Let X be a locally compact Hausdorff space and let $U \subseteq X$ be an open subset with complement $Z = X \setminus U$. If Z and U are totally disconnected, then so is X.*

Proof. As in the proof of Lemma 1.11, we may assume that X is compact and in particular normal. Recall that a compact Hausdorff space is totally disconnected if and only if it has covering dimension zero, e.g. by [Eng78, Thm 3.1.30]. The result follows from the sum theorem for the covering dimension of a normal space [Eng78, Lem 3.1.6], with $K_1 = Z$ and $K_2 = U$ in the notation used there. $\qquad\square$

2. Smooth spaces are discrete

Using the result from the previous section, the question of smoothness for categories of spectral sheaves on locally compact Hausdorff spaces is now reduced to a question about totally disconnected spaces. It is not hard to reduce further to considering compact totally disconnected spaces. For these spaces, derived categories of sheaves admit very convenient descriptions in terms of the algebra of continuous integer-valued functions. Using this description, we can give a simple proof of the main theorem, which states that a locally compact space which is smooth in the sense of Kontsevich must be finite.

---

[7]Recall that for a finite poset P, the *Krull dimension* of P is the largest integer $d \geq 0$ so that there is a strictly increasing chain $p_0 \lneq p_1 \lneq \cdots \lneq p_d$ of length $d + 1$ in P.

2.1. **Sheaves on Stone spaces.** Recall that a *Stone space* is a totally disconnected compact Hausdorff space. It is well-known that the derived category of a Stone space X admits a simple description:

Proposition 2.1. *Let X be a Stone space. There is a canonical equivalence*

$$\mathrm{Shv}(X, \mathcal{D}(\mathbb{Z})) \simeq \mathcal{D}(C(X, \mathbb{Z})),$$

where $C(X, \mathbb{Z})$ is the ring of continuous integer-valued functions on X. Furthermore, under this equivalence the coevaluation

$$\mathcal{D}(\mathbb{Z}) \to \mathcal{D}(C(X \times X, \mathbb{Z}))$$

sends the generator $\mathbb{Z}$ to $C(X, \mathbb{Z})$, viewed as a $C(X \times X, \mathbb{Z})$-module via the ring homomomorphism $\Delta^*\colon C(X \times X, \mathbb{Z}) \to C(X, \mathbb{Z})$ given by restriction along the diagonal immersion.

Proof. We have a triple of adjoint functors

$$\mathrm{Shv}(X, \mathcal{D}(\mathbb{Z})) \xrightarrow{\ \ \Gamma_* \ \ } \mathcal{D}(\mathbb{Z}).$$

We will show that (1) Γ_* is conservative and (2) $\Gamma_* \mathbb{Z}_X \simeq C(X, \mathbb{Z})$, or in other words that the derived global sections of the constant sheaf $\mathbb{Z}_X$ can be computed by taking the underived global sections. Since Γ^* is symmetric monoidal and Γ_* satisfies a projection formula, it will then follow from general nonsense that the canonical induced functor

$$\mathrm{Shv}(X, \mathcal{D}(\mathbb{Z})) \to \mathrm{Mod}_{\Gamma_* \mathbb{Z}_X}(\mathcal{D}(\mathbb{Z})) \simeq \mathcal{D}(C(X, \mathbb{Z}))$$

is an equivalence [MNN17, Prop 5.29].

To show claim (1), suppose $\mathscr{F} \in \mathrm{Shv}(X, \mathcal{D}(\mathbb{Z}))$ has $\mathscr{F}(X) \simeq 0$. Given any clopen subset $U \subseteq X$, the sheaf condition then gives $0 \simeq \mathscr{F}(X) \simeq \mathscr{F}(U) \times \mathscr{F}(X \setminus U)$, and hence $\mathscr{F}(U) \simeq 0$. Since X admits a basis of clopen subsets, it follows that $\mathscr{F} \simeq 0$.[8]

To prove claim (2), it will suffice to show that the classical constant sheaf $\mathbb{Z}_X \in \mathrm{Shv}(X, \mathrm{Ab})$ is Γ_*-acyclic. Recall that a sheaf is *soft* if any section over a closed subspace extends to a global section. On paracompact Hausdorff spaces, soft sheaves are Γ_*-acyclic [God58, Thm II.3.5.2], and hence it will be enough to show that $\mathbb{Z}_X$ is soft. For this, suppose f is a continuous integer-valued function defined in a neighborhood U of a compact subset $K \subseteq X$. Using that X is totally disconnected, we can pick a compact open V with $K \subseteq V \subseteq U$. Let $\psi\colon X \to \mathbb{Z}$ be a continuous integer-valued function with $\psi|_K \equiv 1$ and $\psi|_{X \setminus V} \equiv 0$, as provided by Lemma 2.2 below. Then ψf extends to a continuous integer-valued function on X. $\qquad\square$

Lemma 2.2. *Let X be a totally disconnected locally compact Hausdorff space, and let Z and Z' be disjoint closed subsets of X. If Z is compact, then there is a continuous function $\psi\colon X \to \mathbb{Z}$ such that $\psi|_Z \equiv 0$ and $\psi|_{Z'} \equiv 1$.*

Proof. Equivalently, we must show that there is a clopen neighborhood $U \supseteq Z$ with $U \cap Z' = \varnothing$. This is a straightforward compactness argument. Given $z \in Z$, the fact that X is totally disconnected and locally compact implies that we can find a clopen U_z with $z \in U_z \subseteq X \setminus Z'$. Since $Z \subseteq \bigcup_{z \in Z} U_z$, there must be a finite subset $S \subseteq Z$ with $Z \subseteq \bigcup_{z \in S} U_z$. Then $U = \bigcup_{z \in S}$ is as desired. $\qquad\square$

2.2. **Proof of Theorem 0.2.** We are now ready to prove the main result, by base-changing to $\mathcal{D}(\mathbb{Z})$ and using the model for the derived category described in the previous section. One checks that smoothness is stable under base change.

---

[8]We learned this elementary argument from Ramzi.

Lemma 2.3. *Let X be a totally disconnected locally compact Hausdorff space and let $C(X,\mathbb{Z})$ denote the ring of continuous integer-valued functions. If $V \subseteq X$ is a compact subset that the ideal*

$$I = \{f \in C(X,\mathbb{Z}) \mid f(v) = 0 \text{ for each } v \in V\}$$

is finitely generated, then V is open.

Proof. Recall that a function $X \to \mathbb{Z}$ is continuous if and only if it is locally constant. Assume that $I = (f_1, \ldots, f_n)$. Put $f = \sum_1^n |f_i|$. We claim that $f(x) \neq 0$ for each $x \notin V$. Indeed, for each such x, Lemma 2.2 provides a function $\psi \colon X \to \mathbb{Z}$ with $\psi|_V \equiv 0$ and $\psi(x) = 1$. Hence $\psi \in I$, and it follows that $f_i(x) \neq 0$ for some i, which implies the claim. Hence $V = f^{-1}(0)$, and f is locally constant so V is open as desired. $\square$

Lemma 2.4. *Let X be a topological space and let $\Delta(X) = \{(x,x) \mid x \in X\} \subseteq X \times X$ denote the diagonal. Then X is discrete if and only if $\Delta(X)$ is open.*

Proof. If X is discrete then so is $X \times X$, and hence $\Delta(X)$ is in particular open. On the other hand, suppose $\Delta(X)$ is open. For each $x \in X$, the point $(x,x) \in \Delta(X)$ must then admit an open neighborhood contained in $\Delta(X)$. Using the definition of the product topology, we can then find a smaller open neighborhood of the form $U \times V$, where U and V are open neighborhoods of x in X. But $U \times V \subseteq \Delta(X)$ implies that $U = V = \{x\}$, completing the proof. $\square$

Proposition 2.5. *Let X be a $\mathcal{D}(\mathbb{Z})$-hypercomplete locally compact Hausdorff space such that $\mathrm{Shv}(X, \mathcal{D}(\mathbb{Z}))$ is smooth as a $\mathbb{Z}$-linear stable ∞-category. Then X is finite and discrete.*

Proof. Note that X must be totally disconnected by Lemma 1.1. We first claim that any compact open $U \subseteq X$ will again have that $\mathrm{Shv}(U, \mathcal{D}(\mathbb{Z}))$ is smooth as a $\mathbb{Z}$-linear stable ∞-category. Indeed, let $j \colon U \hookrightarrow X$ be the inclusion. There is a commutative diagram

$$
\begin{array}{ccc}
 & X & \xrightarrow{\Delta_X} X \times X \\
 & \Big\uparrow{\scriptstyle j} & \qquad \Big\uparrow{\scriptstyle j \times j} \\
* \xleftarrow[\Gamma_U]{\Gamma_X} U & \xrightarrow{\Delta_U} & U \times U
\end{array}
$$

and thus

$$
\begin{aligned}
\Delta_{U,*}\Gamma_U^! &\simeq \Delta_{U,*}j^!\Gamma_X^! \\
&\simeq \Delta_{U,*}j^*\Gamma_X^! \\
&\simeq (j \times j)^*\Delta_{X,*}\Gamma_X^!,
\end{aligned}
$$

where the second equivalence uses that j is an open embedding and the last equivalence comes from proper base change. But smoothness of $\mathrm{Shv}(X, \mathcal{D}(\mathbb{Z}))$ means that $\Delta_{X,*}\Gamma_X^!$ is strongly continuous, and $(j \times j)^*$ is strongly continuous since j is proper, whence the composition is also strongly continuous, and hence $\mathrm{Shv}(U, \mathcal{D}(\mathbb{Z}))$ is smooth. By Proposition 2.1, this implies that $C(U, \mathbb{Z})$ is a perfect $C(U \times U, \mathbb{Z})$-module. In particular, the epimorphism $\Delta^* \colon C(U \times U, \mathbb{Z}) \to C(U, \mathbb{Z})$ must have finitely generated kernel. But this kernel is exactly $I = \{f \in C(U \times U) \mid f(u,u) = 0 \text{ for each } u \in U\}$, whence the diagonal $\Delta(U) \subseteq U \times U$ must be open according to Lemma 2.3. But Lemma 2.4 then implies that U is discrete. Since being discrete is a local property, we conclude that X is discrete.

As X is discrete, we find as in Remark 0.4 that

$$
\Delta_*\Gamma^!\mathbb{Z}_{(x,y)} \simeq
\begin{cases}
\mathbb{Z}, & \text{if } x = y, \\
0, & \text{else.}
\end{cases}
$$

In particular $\mathrm{supp}\,\Delta_*\Gamma^!\mathbb{Z} = \Delta(X)$, and then by [Har23, Thm 2.3] $\Delta(X) \cong X$ must be compact, but this implies that X is finite. $\square$

Proof of Theorem 0.2. If $\mathrm{Shv}(X, \mathrm{Sp})$ is smooth, then $\mathrm{Shv}(X, \mathrm{Sp}) \otimes \mathcal{D}(\mathbb{Z}) \simeq \mathrm{Shv}(X, \mathcal{D}(\mathbb{Z}))$ is smooth as a $\mathbb{Z}$-linear category, and the result therefore follows from the previous proposition. $\square$

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