# COMPACT SHEAVES ON A LOCALLY COMPACT SPACE

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Abstract. We describe the compact objects in the $\infty$-category of $\mathcal{C}$-valued sheaves $\mathrm{Shv}(X, \mathcal{C})$ on a hypercomplete locally compact Hausdorff space $X$, for $\mathcal{C}$ a compactly generated stable $\infty$-category. When $X$ is a non-compact connected manifold and $\mathcal{C}$ is the unbounded derived category of a ring, our result recovers a result of Neeman. Furthermore, for $X$ as above and $\mathcal{C}$ a nontrivial compactly generated stable $\infty$-category, we show that $\mathrm{Shv}(X, \mathcal{C})$ is compactly generated if and only if $X$ is totally disconnected.

The aim of this note is to clarify and expand on a point made by Neeman [Nee01]. Let $M$ be a non-compact connected manifold, and let $\mathrm{Shv}(M, \mathcal{D}(\mathbb{Z}))$ denote the unbounded derived category of sheaves of abelian groups on $M$. Neeman shows that, up to equivalence, the only compact object in $\mathrm{Shv}(M, \mathcal{D}(\mathbb{Z}))$ is the zero sheaf. This implies that $\mathrm{Shv}(M, \mathcal{D}(\mathbb{Z}))$ is very far from compactly generated. Nevertheless, it follows from Lurie's covariant Verdier duality theorem [Lur17, Thm 5.5.5.1] that $\mathrm{Shv}(M, \mathcal{D}(\mathbb{Z}))$ satisfies a related smallness condition: it is *dualizable* in the symmetric monoidal $\infty$-category $\mathcal{P}\mathrm{r}^{\otimes}_{\mathrm{stab}}$ of stable presentable $\infty$-categories, which holds more generally if $M$ is replaced with any locally compact Hausdorff space $X$. Although every compactly generated presentable stable $\infty$-category is dualizable [Lur18, Prop D.7.2.3], Neeman's example thus shows that the converse is false. The existence of this large and interesting class of stable presentable $\infty$-categories that are dualizable but not compactly generated forms part of the motivation behind Efimov's continuous extensions of localizing invariants [Efi22], see also [Hoy18].

This note is concerned with the following two questions about the $\infty$-category of $\mathcal{C}$-valued sheaves on a general locally compact Hausdorff space $X$, where $\mathcal{C}$ is some compactly generated stable $\infty$-category (e.g. the unbounded derived $\infty$-category of a ring or the $\infty$-category of spectra):

(1) How rare is it for $\mathrm{Shv}(X, \mathcal{C})$ to be compactly generated?
(2) How far is $\mathrm{Shv}(X, \mathcal{C})$ from being compactly generated in general?

With a relatively mild completeness assumption on $X$ (see Section 1), we answer question (2) by showing that a $\mathcal{C}$-valued sheaf $\mathscr{F}$ on $X$ is compact as an object of $\mathrm{Shv}(X, \mathcal{C})$ if and only if it has compact support, compact stalks, and is locally constant (Theorem 2.3).[1] Thus if $X$ is for instance a CW complex, the subcategory of compact objects $\mathrm{Shv}(X, \mathcal{C})^{\omega}$ depends only on the *homotopy type* of the compact path components of $X$, and it is therefore impossible to reconstruct the entire sheaf category $\mathrm{Shv}(X, \mathcal{C})$ from this information.

In his 2022 ICM talk, Efimov mentions that the $\infty$-category of $\mathcal{D}(R)$-valued sheaves on a locally compact Hausdorff space $X$ 'is almost never compactly generated (unless $X$ is totally disconnected, like a Cantor set)' [Efi22, slide 13]. As a corollary to our description of the compact objects of $\mathrm{Shv}(X, \mathcal{C})$, we verify–modulo the same completeness assumption mentioned above– that indeed the *only* locally compact Hausdorff spaces $X$ with $\mathrm{Shv}(X, \mathcal{C})$ compactly generated, for some nontrivial $\mathcal{C}$, are the totally disconnected ones (Proposition 3.1), thereby answering question (1).

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[1]Since posting this note on the arXiv, we became aware that Scholze has indicated a proof of this statement for $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ in his notes on six-functor formalisms [Sch, Prop 7.11]. The approach taken there, which uses descent to deduce the general statement from the case where $X$ is a profinite set, is different from the one we take.

**Notation and conventions.** Throughout this note, we use the theory of higher categories and higher algebra, an extensive textbook account of which can be found in the work of Lurie [Lur09, Lur17, Lur18]. We will also make frequent use of the six-functor formalism for derived sheaves on locally compact Hausdorff spaces, as described classically by [Ver65] and with general coefficients by [Vol23].

For convenience, we assume the existence of an uncountable Grothendieck universe $\mathcal{U}$ of *small* sets and further Grothendieck universes $\mathcal{U}'$ and $\mathcal{U}''$ of *large* and *very large* sets respectively, such that $\mathcal{U} \in \mathcal{U}' \in \mathcal{U}''$. 'Topological space' always implicitly refers to a small topological space, and similarly with 'spectrum'. On the other hand, an '$\infty$-category' is an '$(\infty, 1)$-category' is a quasicategory, which unless otherwise stated is large. We let $\widehat{\mathcal{C}\mathrm{at}}_\infty$ denote the very large $\infty$-category of (large) $\infty$-categories.

Because we are dealing with sheaves on topological spaces, we deem it best to make a clear distinction between actual topological spaces on the one hand, and on the other hand the objects of the $\infty$-category $\mathcal{S}$ of 'spaces' in the sense of Lurie. Following the convention suggested in [CS23], we will refer to the latter as *anima*.

Given an $\infty$-category $\mathcal{C}$, we let $\mathcal{C}^\omega \subseteq \mathcal{C}$ denote the subcategory spanned by the compact objects. Recall that an object $C \in \mathcal{C}$ is said to be *compact* if the presheaf of large anima $D \mapsto \mathrm{Map}_{\mathcal{C}}(C, D)$ preserves small filtered colimits.

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## 1. $\mathcal{C}$-hypercomplete spaces

Given an $\infty$-category $\mathcal{C}$ and a topological space $X$, we let $\mathrm{Shv}(X, \mathcal{C})$ denote the $\infty$-category of $\mathcal{C}$-valued sheaves on $X$ in the sense of Lurie [Lur09]. That is, $\mathrm{Shv}(X, \mathcal{C})$ is the full subcategory of the presheaf $\infty$-category $\mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathcal{C})$ consisting of presheaves $\mathscr{F}$ satisfying the *sheaf condition*: for any open set $U \subseteq X$ and any open cover $\{U_i \to U\}_{i \in I}$, the canonical map

$$\mathscr{F}(U) \to \lim_V \mathscr{F}(V)$$

is an equivalence, where $V$ ranges over open sets $V \subseteq U_i \subseteq X$, $i \in I$, considered as a poset under inclusion. When $\mathcal{C} = \mathcal{S}$ is the $\infty$-category of anima, we will abbreviate $\mathrm{Shv}(X) = \mathrm{Shv}(X, \mathcal{S})$.

**Remark 1.1.** When $\mathcal{C} = \mathcal{D}(R)$ is the unbounded derived $\infty$-category of a ring, the $\infty$-category $\mathrm{Shv}(X, \mathcal{D}(R))$ is related to, but generally not the same as, the derived $\infty$-category $\mathcal{D}(\mathrm{Shv}(X, R))$ of the ordinary category of sheaves of $R$-modules on $X$, which is the object studied (via its homotopy category) by Neeman [Nee01]. However, they do coincide under the completeness assumption that we will impose on $X$ below, see [Sch, Prop 7.1]. Since this completeness assumption is verified when $X$ is a topological manifold, our results include those of Neeman.

We are interested in topological spaces satisfying the following condition:

**Definition 1.2.** A topological space $X$ is *$\mathcal{C}$-hypercomplete* if the stalk functors $x^* \colon \mathrm{Shv}(X, \mathcal{C}) \to \mathcal{C}$ are jointly conservative for $x$ ranging over the points of $X$.

The reason for our choice of terminology is that $X$ is $\mathcal{S}$-hypercomplete if and only if the 0-localic $\infty$-topos $\mathrm{Shv}(X)$ has enough points, which is equivalent to $\mathrm{Shv}(X)$ being hypercomplete as an $\infty$-topos by Claim (6) in [Lur09, § 6.5.4]. (This is *not* true for arbitrary $\infty$-topoi, i.e. there are hypercomplete $\infty$-topoi that do not have enough points.) This subtlety, whereby a morphism of sheaves may fail to be an equivalence even though it is so on all stalks, does not occur for non-derived sheaves: the 1-topos $\mathrm{Shv}(X, \mathcal{S}_{\leq 0})$ of sheaves of sets on a topological space $X$ *always* has enough points. We refer to [Lur09, § 6.5.4] for a discussion of why it is often preferable in the homotopical setting to work with non-hypercomplete sheaves, rather than, say, imposing hypercompleteness by replacing $\mathrm{Shv}(X)$ with its hypercompletion $\mathrm{Shv}(X)^\wedge$.

The following observation provides us with a source of $\mathcal{C}$-hypercomplete spaces:

**Proposition 1.3.** *Let $X$ be an $\mathcal{S}$-hypercomplete topological space. Then $X$ is also $\mathcal{C}$-hypercomplete for any compactly generated $\infty$-category $\mathcal{C}$.*

*Proof.* Let $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^\omega, \mathrm{Shv}(X)) \subseteq \mathrm{Fun}(\mathcal{C}^\omega, \mathrm{Shv}(X))$ denote the full subcategory spanned by left exact functors, i.e. functors that preserve finite limits. The statement then follows from the fact that there is a natural equivalence

$$\mathrm{Shv}(X, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^\omega, \mathrm{Shv}(X)) \tag{1}$$

which is given informally by sending a sheaf $\mathscr{F}$ to the functor

$$C \mapsto [U \mapsto \mathrm{Map}_{\mathcal{C}}(C, \mathscr{F}(U))],$$

see [ØJ22, Lem B.3]. □

The literature describes several classes of topological spaces that are $\mathcal{S}$-hypercomplete. Here is a list of some classes of topological spaces that have this property:

- paracompact spaces that are locally of covering dimension $\leq n$ for some fixed $n$ [Lur09, Cor 7.2.1.12],
- arbitrary CW complexes [Hoy16],
- finite-dimensional Heyting spaces [Lur09, Rem 7.2.4.18], and
- Alexandrov spaces, since the $\infty$-topos of sheaves associated to an Alexandrov space is equivalent to a presheaf $\infty$-topos.

## 2. When is a sheaf compact?

Let $\mathcal{C}$ be a compactly generated stable $\infty$-category, e.g. the unbounded derived category $\mathcal{D}(R)$ of a ring $R$ or the $\infty$-category of spectra Sp. Given a sheaf $\mathscr{F} \in \mathrm{Shv}(X, \mathcal{C})$, we define the *support* of $\mathscr{F}$ by

$$\mathrm{supp}\,\mathscr{F} = \{x \in X \mid \mathscr{F}_x \not\simeq 0\} \subseteq X.$$

As in [Nee01], our study of the compact objects of $\mathrm{Shv}(X, \mathcal{C})$ proceeds from an analysis of their supports. By slightly adapting the proof of [Nee01, Lem 1.4], we get the following description of the support of a compact sheaf:

**Lemma 2.1.** *Let $X$ be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space and let $\mathscr{F} \in \mathrm{Shv}(X, \mathcal{C})^\omega$. Then the support $\mathrm{supp}\,\mathscr{F}$ is compact.*

*Proof.* We first show that $\mathrm{supp}\,\mathscr{F}$ is contained in a compact subset of $X$. Consider the canonical map

$$\mathrm{colim}_U (j_U)_! j_U^* \mathscr{F} \to \mathscr{F}, \tag{2}$$

where the colimit ranges over the poset of precompact open sets ordered by the rule $U \leq V$ if $\overline{U} \subseteq V$, and for each such $U$ we have denoted by $j_U\colon U \hookrightarrow X$ the inclusion. Since precompact open sets form a basis for the topology on $X$, the map (2) is an equivalence of sheaves. Let $\phi\colon \mathscr{F} \xrightarrow{\sim} \mathrm{colim}_U (j_U)_!(j_U)^*\mathscr{F}$ be some choice of inverse. Any finite union of precompact open sets is again precompact open, so the poset of precompact open sets is filtered. Hence compactness of $\mathscr{F}$ implies that $\phi$ factors through $(j_U)_! j_U^*\mathscr{F}$ for some precompact open $U$, and it follows that $\mathrm{supp}\,\mathscr{F}$ is contained in a compact subset $\overline{U} \subseteq X$, as claimed.

By the above, it remains only to be seen that $\mathrm{supp}\,\mathscr{F}$ is closed, or equivalently that its complement $X \setminus \mathrm{supp}\,\mathscr{F}$ is open. Suppose $x \in X \setminus \mathrm{supp}\,\mathscr{F}$. Then we have a recollement fiber sequence

$$j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F},$$

where $j\colon X \setminus \{x\} \hookrightarrow X$ and $i\colon \{x\} \hookrightarrow X$ are the inclusions, and since $x \notin \mathrm{supp}\,\mathscr{F}$ we have $j_! j^* \mathscr{F} \simeq \mathscr{F}$. Since $j_!$ is a fully faithful left adjoint, it reflects compact objects, and we conclude that $j^*\mathscr{F}$ is again compact. But then $j^*\mathscr{F}$ is supported on a compact subset of $X \setminus \{x\}$ by the above, which must be closed as a subset of $X$, and hence $x$ lies in the interior of $X \setminus \mathrm{supp}\,\mathscr{F}$ as desired. □

**Lemma 2.2.** *If $f\colon X \to Y$ is a proper map of locally compact Hausdorff spaces, then the pullback functor $f^*$ preserves compact objects. In particular, if $X$ is a compact Hausdorff space and $E \in \mathcal{C}^\omega$, then $E_X \in \mathrm{Shv}(X,\mathcal{C})^\omega$, where $E_X$ denotes the constant sheaf at $E$.*

*Proof.* Since $f$ is proper, the pullback $f^*$ is left adjoint to $f_* \simeq f_!$, which is itself left adjoint to $f^!$. Hence $f_*$ preserves colimits, and it follows that its left adjoint $f^*$ preserves compact objects. The statement about constant sheaves follows by taking $f$ to be the projection from $X$ to a point. □

Our main result is the following description of the compact objects in $\mathrm{Shv}(X,\mathcal{C})$:

**Theorem 2.3.** *Let $X$ be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space. A sheaf $\mathscr{F} \in \mathrm{Shv}(X,\mathcal{C})$ is compact if and only if*

*(i)* $\mathrm{supp}\,\mathscr{F}$ *is compact;*
*(ii)* $\mathscr{F}$ *is locally constant; and*
*(iii)* $\mathscr{F}_x \in \mathcal{C}^\omega$ *for each* $x \in X$.

In particular, note that conditions (i) and (ii) together imply that if $\mathscr{F}$ is compact, then the support $\mathscr{F}$ must be a compact open subset of $X$. On the other hand, (iii) guarantees that if $\mathscr{F}$ is constant on $U \subseteq X$, say with value $E$, then $E \in \mathcal{C}^\omega$.

*Proof.* 'Necessity.' Suppose we are given $\mathscr{F} \in \mathrm{Shv}(X,\mathcal{C})^\omega$. Then $\mathscr{F}$ must satisfy (i) by Lemma 2.1 and (ii) by Lemma 2.2, since the stalk $\mathscr{F}_x$ at $x \in X$ is the same as the pullback $i_x^*\mathscr{F}$ along the inclusion $i_x\colon \{x\} \hookrightarrow X$, which is always a proper map. It remains only to be seen that $\mathscr{F}$ is locally constant. Fix a point $x \in X$, and let $i_x$ again denote the inclusion of this point into $X$. Let $E = i_x^*\mathscr{F}$ denote the stalk of $\mathscr{F}$ at $x$. By [Lur09, Cor 7.1.5.6], there is an equivalence $\mathrm{colim}_U\, \mathscr{F}(U) \simeq E$, where $U$ ranges over the poset of open neighborhoods of $x$. As $E$ is compact, this implies that $\mathscr{F}(U) \to E$ has a section for some $U$. Pick a precompact open neighborhood $W \ni x$ with $\overline{W} \subseteq U$, and let $i\colon \overline{W} \hookrightarrow X$ denote the inclusion. As the canonical map $\mathscr{F}(U) \to E$ factors through the restriction $\mathscr{F}(U) \to (i^*\mathscr{F})(\overline{W}) \to \mathscr{F}(W)$, the map $(i^*\mathscr{F})(\overline{W}) \to E$ also admits a section $E \to (i^*\mathscr{F})(\overline{W})$. Viewing the latter as a morphism from the constant presheaf with value $E$ to $i^*\mathscr{F}$, we get an induced map $\sigma\colon E_{\overline{W}} \to i^*\mathscr{F}$ of sheaves over $\overline{W}$ which by construction induces an equivalence of stalks at $x$. Here both $E_{\overline{W}}$ and $i^*\mathscr{F}$ are compact, so the cofiber $\mathscr{Q} = \mathrm{cofib}(\sigma)$ is again compact. But then $\mathrm{supp}\,\mathscr{Q}$ is compact, so $W' = W \setminus \mathrm{supp}\,\mathscr{Q}$ is open and $\mathscr{Q}_x \simeq 0$ so $x \in W'$. Furthermore, $\sigma$ restricts to an equivalence

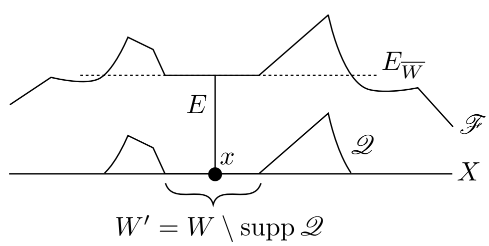


FIGURE 1. 'Espace étale' visualization of the fiber sequence $E_{\overline{W}} \to \mathscr{F} \to \mathscr{Q}$

of sheaves on $W'$ by construction, so $\mathscr{F}|_{W'}$ is equivalent to the constant sheaf on $W'$ with value $E$, as desired.

'Sufficiency.' Let $i\colon \mathrm{supp}\,\mathscr{F} \hookrightarrow X$ denote the inclusion. Since $i$ is both proper and an open immersion, both $i_* \simeq i_!$ and $i^* \simeq i^!$ preserve and reflect compact objects. We may therefore assume that $X$ is compact, after possibly replacing it with $\mathrm{supp}\,\mathscr{F}$. Pick a finite collection of closed subsets $Z_i \subseteq X$, $i = 1, \ldots, n$, such that $\mathscr{F}$ is constant in a neighborhood of each $Z_i$ and such that $X$ is covered by the interiors $Z_i^\circ$. Descent (Corollary A.3) implies that the canonical functor

$$\begin{aligned}&\mathrm{Shv}(X,\mathcal{C})\\ &\to \lim_{\Delta_{\leq n-1}}\left( \mathrm{Shv}(\textstyle\bigcap_1^n Z_i,\mathcal{C}) \;\substack{\longrightarrow\\ \vdots\\ \longrightarrow}\; \cdots \;\substack{\longrightarrow\\ \leftarrow\\ \longrightarrow\\ \leftarrow\\ \longrightarrow}\; \prod_{i,j}\mathrm{Shv}(Z_i\cap Z_j,\mathcal{C}) \;\substack{\longrightarrow\\ \leftarrow\\ \longrightarrow}\; \prod_i \mathrm{Shv}(Z_i,\mathcal{C}) \right)\end{aligned}$$

is an equivalence. Write $I = \{1,\dots,n\}$ for short and put $Z_J = \bigcap_{j\in J} Z_j$ for each $J\subseteq I$. The canonical projection from $\mathrm{Shv}(X,\mathcal{C})$ to $\mathrm{Shv}(Z_J,\mathcal{C})$ is the restriction map. By construction, we have that for each $J\subseteq I$, the restriction $\mathscr{F}|_{Z_J}$ is constant with value a compact object, and hence compact as an object of $\mathrm{Shv}(Z_J,\mathcal{C})$ by the preceding lemma. According to [Lur09, Lem 6.3.3.6], the identity functor $\mathrm{id}\colon \mathrm{Shv}(X,\mathcal{C})\to\mathrm{Shv}(X,\mathcal{C})$ is the colimit of a diagram $\Delta_{\leq n-1}\to \mathrm{Fun}(\mathrm{Shv}(X,\mathcal{C}),\mathrm{Shv}(X,\mathcal{C}))$ which sends the object $[k]\in\Delta_{\leq n-1}$ to the composition

$$\mathrm{Shv}(X,\mathcal{C}) \xrightarrow{\;i_k^*\;} \prod_{\substack{J\subseteq I,\\ |J|=k}} \mathrm{Shv}(Z_J,\mathcal{C}) \simeq \mathrm{Shv}\Bigg(\coprod_{\substack{J\subseteq I,\\ |J|=k}} Z_J,\mathcal{C}\Bigg) \xrightarrow{(i_k)_*} \mathrm{Shv}(X,\mathcal{C}),$$

and so for any filtered system $\{\mathscr{G}_\alpha\}_{\alpha\in A}$, we find

$$\begin{aligned}\mathrm{Map}(\mathscr{F},\mathrm{colim}_A\,\mathscr{G}_\alpha) &\simeq \lim_{[k]\in\Delta_{\leq n-1}} \mathrm{Map}(\mathscr{F},(i_k)_*i_k^*\,\mathrm{colim}\,\mathscr{G}_\alpha)\\ &\simeq \lim_{[k]\in\Delta_{\leq n-1}} \mathrm{Map}(i_k^*\mathscr{F},\mathrm{colim}_A\, i_k^*\mathscr{G}_\alpha)\\ &\simeq \lim_{[k]\in\Delta_{\leq n-1}} \mathrm{colim}_A\,\mathrm{Map}(i_k^*\mathscr{F}, i_k^*\mathscr{G}_\alpha)\\ &\simeq \mathrm{colim}_A \lim_{[k]\in\Delta_{\leq n-1}} \mathrm{Map}(i_k^*\mathscr{F}, i_k^*\mathscr{G}_\alpha)\\ &\simeq \mathrm{colim}_A\,\mathrm{Map}(\mathscr{F},\mathscr{G}_\alpha),\end{aligned}$$

where the third equivalence uses that the restriction $i_k^*\mathscr{F}$ is compact[2] and the second-last equivalence uses that filtered colimits are left exact in $\mathcal{S}$. □

As a corollary, we recover Neeman's result:

**Corollary 2.4** (Neeman)**.** *Let $M$ be a non-compact connected manifold. Then $\mathscr{F}\in\mathrm{Shv}(M,\mathcal{C})^\omega$ if and only if $\mathscr{F}\simeq 0$.*

In fact our result shows that the conclusion of Neeman's result holds more generally if $M$ is replaced by a $\mathcal{C}$-hypercomplete locally compact Hausdorff space $X$ whose quasicomponents are all non-compact.

As a further corollary to our theorem, we will describe the dualizable objects of the category of sheaves on a locally compact Hausdorff space. Suppose that $\mathcal{C}$ has the structure of a presentably monoidal $\infty$-category $\mathcal{C}^\otimes\in\mathrm{Alg}_{\mathbb{E}_1}(\mathcal{P}\mathrm{r}^\otimes_{\mathrm{stab}})$, meaning roughly that $\mathcal{C}$ has a coherently associative and unital tensor product $\otimes$ that commutes with colimits in each variable. We let $\mathbf{1}\in\mathcal{C}$ denote the unit with respect to $\otimes$. Recall that an object $D\in\mathcal{C}$ is said to be *right dualizable* if there exists an object $D^\vee\in\mathcal{D}$ and a morphism $e\colon D^\vee\otimes D\to\mathbf{1}$ such that for all $E,F\in\mathcal{C}$, the map

$$\mathrm{Map}_{\mathcal{C}}(E,F\otimes D^\vee)\xrightarrow{-\otimes D}\mathrm{Map}_{\mathcal{C}}(E\otimes D,F\otimes D^\vee\otimes D)\xrightarrow{(F\otimes e)\circ}\mathrm{Map}_{\mathcal{C}}(E\otimes D,F) \tag{3}$$

is an equivalence. Right dualizability is an algebraic smallness condition, just as compactness is a purely categorical smallness condition. Indeed, if the unit $\mathbf{1}$ is compact as an object of $\mathcal{C}$, then by a well-known observation every right dualizable object of $\mathcal{C}$ is compact. To see this, suppose

[2]Indeed, we have already observed that $\mathscr{F}|_{Z_J}$ is compact for each $J$, and hence the associated object $i_k^*\mathscr{F}$ in the product $\Pi_J\,\mathrm{Shv}(Z_J,\mathcal{C})$is also compact according to [Lur09, Lem 5.3.4.10].

$D \in \mathcal{C}$ is right dualizable and $I \to \mathcal{C}$, $i \mapsto E_i$, is a filtered diagram of objects in $\mathcal{C}$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Map}_{\mathcal{C}}(D, \mathrm{colim}_I\, E_i) & \longrightarrow & \mathrm{colim}_I\, \mathrm{Map}_{\mathcal{C}}(D, E_i) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{Map}_{\mathcal{C}}(\mathbf{1} \otimes D, \mathrm{colim}_I\, E_i) & \longrightarrow & \mathrm{colim}_I\, \mathrm{Map}_{\mathcal{C}}(\mathbf{1} \otimes D, E_i) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, (\mathrm{colim}_I\, E_i) \otimes D^\vee)) & \longrightarrow & \mathrm{colim}_I\, \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, E_i \otimes D^\vee) \\
& \searrow^{\simeq} \qquad {}^{\simeq}\nearrow & \\
& \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, \mathrm{colim}_I(E_i \otimes D^\vee)) &
\end{array}$$

where the vertical maps in the top square are induced by the unit equivalences $D \simeq \mathbf{1} \otimes D$, the vertical maps in the second square are the equivalences of the form (3) coming from the assumption that $D$ is dualizable, and the lower triangle shows that the lowest straight horizontal arrow factors as post-composition with the canonical equivalence

$$(\mathrm{colim}_I\, E_i) \otimes D^\vee) \simeq \mathrm{colim}_I(E_i \otimes D^\vee),$$

where we use that $\otimes$ preserves colimits, followed by the canonical map

$$\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, \mathrm{colim}_I(E_i \otimes D^\vee)) \to \mathrm{colim}_I\, \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, E_i \otimes D^\vee),$$

which we know to be an equivalence by our assumption that $\mathbf{1}$ is compact.

Given a presentably monoidal stable $\infty$-category $\mathcal{C}^\otimes$ as above and a topological space $X$, one can equip also the $\infty$-category of $\mathcal{C}$-valued sheaves $\mathrm{Shv}(X, \mathcal{C})$ with the structure of a presentably monoidal $\infty$-category, which is roughly given by defining the tensor product of $\mathscr{F}, \mathscr{G} \in \mathrm{Shv}(X, \mathcal{C})$ to be the sheafification of the presheaf

$$U \mapsto \mathscr{F}(U) \otimes \mathscr{G}(U).$$

(For a precise definition, see e.g. the discussion following [Vol23, Thm 1.3].) The unit with respect to this tensor product is the constant sheaf $\mathbf{1}_X$ at the unit $\mathbf{1} \in \mathcal{C}$, and for each continuous map $f\colon Y \to X$, the pullback $f^*\colon \mathrm{Shv}(X, \mathcal{C}) \to \mathrm{Shv}(Y, \mathcal{C})$ can be canonically endowed with the structure of a monoidal functor. In a similar vein to the question answered by Theorem 2.3, one could ask for a classification of the dualizable objects of $\mathrm{Shv}(X, \mathcal{C})^\otimes$ with respect to the monoidal structure defined above, when $X$ is a $\mathcal{C}$-hypercomplete locally compact Hausdorff space. It turns out that it is now straightforward to answer this question:

**Corollary 2.5.** *Let $\mathcal{C}^\otimes$ be a presentably monoidal stable $\infty$-category, whose underlying $\infty$-category is compactly generated and such that the unit $\mathbf{1} \in \mathcal{C}$ is compact. Let $X$ be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space. With respect to the induced symmetric monoidal structure on* $\mathrm{Shv}(X, \mathcal{C})$*, a sheaf $\mathscr{F} \in \mathrm{Shv}(X, \mathcal{C})$ is right dualizable if and only if*

*(i) $\mathscr{F}$ is locally constant, and*
*(ii) $\mathscr{F}_x \in \mathcal{C}$ is right dualizable for each $x \in X$.*

*Proof.* 'Sufficiency.' Let $\mathscr{F}$ be a sheaf satisfying conditions (i) and (ii), and let $\mathcal{U}$ be an open cover of $X$ such that $\mathscr{F}|_U$ is equivalent to a constant sheaf for each $U \in \mathcal{U}$. Čech descent implies that $\mathrm{Shv}(X, \mathcal{C})$ is equivalent to the limit $\lim_V \mathrm{Shv}(V, \mathcal{C})$, as $V$ runs over the poset of open sets $V$ such that $V \subseteq U$ for some $U \in \mathcal{U}$. For each of these $V$ we have that $\mathscr{F}|_V$ is equivalent to a constant sheaf, which if $V \neq \varnothing$ will be of the form $\pi^* \mathscr{F}_x$, where $\pi\colon V \to *$ is the projection and $\mathscr{F}_x$ is the stalk at any $x \in V$. But $\pi^*$ is monoidal and hence preserves right dualizable objects, whence we by dualizability of $\mathscr{F}_x$ know that $\mathscr{F}|_V$ is right dualizable too. It now follows from the descent property for dualizability [Lur17, Prop 4.6.1.11] that $\mathscr{F}$ is right dualizable as an object of $\mathrm{Shv}(X, \mathcal{C})$.

'Necessity.' Assume that $\mathscr{F}$ is right dualizable, and let $x \in X$ be some point. The condition on the stalks is immediate, since pullback preserves dualizable sheaves. We must show that $\mathscr{F}$

is locally constant in a neighborhood of $x$. Pick a precompact open neighborhood $U \ni x$. Then $\mathscr{F}|_{\overline{U}}$ is again right dualizable, and since the monoidal unit $\mathbf{1}_{\overline{U}} \simeq \pi^*\mathbf{1} \in \mathrm{Shv}(\overline{U}, \mathcal{C})$ is compact, it follows that $\mathscr{F}|_{\overline{U}}$ is compact as an object of $\mathrm{Shv}(\overline{U}, \mathcal{C})$. But then the previous theorem implies that it must be locally constant on $\overline{U}$, and hence also on the subset $U$ as desired. □

The preceding argument gives an analogous classification of the *left* dualizable sheaves, where left dualizability in a presentably monoidal $\infty$-category $\mathcal{C}^{\otimes}$ defined in terms of a morphism $e\colon D \otimes D^{\vee} \to \mathbf{1}$ instead.

## 3. When is $\mathrm{Shv}(X, \mathcal{C})$ compactly generated?

In this section, we prove the following characterization of those locally compact Hausdorff spaces $X$ that have $\mathrm{Shv}(X, \mathcal{C})$ compactly generated:

**Proposition 3.1.** *Let $\mathcal{C}$ be a non-trivial compactly generated stable $\infty$-category, and let $X$ be a $\mathcal{C}$-hypercomplete locally compact Hausdorff space. Then $\mathrm{Shv}(X, \mathcal{C})$ is compactly generated if and only if $X$ is totally disconnected.*

**Remark 3.2.** If $X$ is totally disconnected, then $X$ is automatically $\mathcal{S}$-hypercomplete. This follows from [Lur09, Rmk 7.2.4.18], since $X$ is a zero-dimensional. In particular, this does not require $X$ to be paracompact.

3.1. **Proof of the proposition.** The proof will use the following observation:[3]

**Lemma 3.3.** *Let $\mathcal{C}$ be a compactly generated stable $\infty$-category, and let $\{C_i\}_{i\in I}$ and $\{D_i\}_{i\in I}$ be filtered systems in $\mathcal{C}$ indexed over the same poset $I$.*

(1) *Suppose that for each $i \in I$, there is some $j \geq i$ so that the transition map $C_i \to C_j$ factors through the zero object $*$. Then $\mathrm{colim}_I C_i \simeq *$. If each $C_i$ is compact, then the converse holds.*
(2) *Suppose that for each comparable pair $i \leq j$ in $I$ there are horizontal equivalences making*

$$\begin{array}{ccc} C_i & \overset{\simeq}{\dashrightarrow} & D_i \\ \downarrow & & \downarrow \\ C_j & \overset{\simeq}{\dashrightarrow} & D_j \end{array}$$

*commute, where the vertical maps are the transition maps. If each $C_i$ is compact, then $\mathrm{colim}_I C_i \simeq *$ if and only if $\mathrm{colim}_I D_i \simeq *$.*

*Proof.* Note that (2) follows from (1), since the existence of such commutative squares implies that $\{C_i\}_I$ has the vanishing property for transition maps described in (1) if and only if $\{D_i\}_I$ has that property.

For the first claim in (1), it suffices to show that $\mathrm{Map}_{\mathcal{C}}(D, \mathrm{colim}_{i\in I} C_i)$ is contractible for each compact object $D \in \mathcal{C}^{\omega}$. For this, first observe that

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(D, \mathrm{colim}_{i\in I} C_i) \cong \mathrm{colim}_{i\in I} \pi_0 \mathrm{Map}(D, C_i) \cong *,$$

since our assumption guarantees that any homotopy class $D \to C_i$ is identified $D \to * \to C_i$ after postcomposing with the transition map $C_i \to C_j$ for sufficiently large $j \geq i$. Applying the same argument for the compact object $\Sigma^n D$, $n \geq 1$, we find that

$$\pi_n \mathrm{Map}_{\mathcal{C}}(D, \mathrm{colim}_{i\in I} C_i) \cong \pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^n D, \mathrm{colim}_{i\in I} C_i)$$

vanishes also.

Assume now that each $C_i$ is compact and that $\mathrm{colim}_I C_i \simeq *$. Then

$$\mathrm{colim}_{j\in I} \mathrm{Map}_{\mathcal{C}}(C_i, C_j) \simeq \mathrm{Map}_{\mathcal{C}}(C_i, \mathrm{colim}_{j\in I} C_j),$$

[3] I am thankful to Maxime Ramzi for pointing out that an earlier incarnation of this lemma, which appeared in the first arXiv version of this note, was incorrect. The following proof of the more restricted lemma was suggested to me by Jesper Grodal (and also by Ramzi when he pointed out the error). Fortunately, the arguments in this note only ever required the current version of the lemma.

and since $\pi_0$ commutes with filtered colimits of anima, it follows that for sufficiently large $j \geq i$ the transition map $C_i \to C_j$ is homotopic to $C_i \to * \to C_j$. □

*Proof of Proposition 3.1.* ‘Sufficiency.’ The $\infty$-category of sheaves of anima $\mathrm{Shv}(X)$ is compactly generated by [Lur09, Prop 6.5.4.4], and hence so is $\mathrm{Shv}(X, \mathcal{C}) \simeq \mathrm{Shv}(X) \otimes \mathcal{C}$ according to [Lur17, Lem 5.3.2.11].

‘Necessity.’ Let $x \in X$. We must show that if $y \in X$ lies in the same connected component as $X$, then $y = x$. For this, pick an object $C \not\simeq 0$ in $\mathcal{C}$ and let $x_*C$ denote the skyscraper sheaf at $x$ with value $C$. By assumption there is a filtered system $\{\mathscr{F}_i\}_{i \in I}$ of compact sheaves with $\mathrm{colim}_I\, \mathscr{F}_i \simeq x_*C$. For each $i$, the fact that $\mathscr{F}_i$ is locally constant and that $x$ and $y$ lie in the same connected component means there is a non-canonical equivalence of stalks $x^*\mathscr{F}_i \simeq y^*\mathscr{F}_i$. One should not expect to find a system of such non-canonical equivalences assembling into a natural transformation, essentially because the neighborhoods on which the $\mathscr{F}_i$ are constant could get smaller and smaller as $i$ increases. Nevertheless, given a comparable pair $i \leq j$ in $I$, one can pick equivalences making the diagram

$$\begin{array}{ccc} x^*\mathscr{F}_i & \overset{\simeq}{\dashrightarrow} & y^*\mathscr{F}_i \\ \downarrow & & \downarrow \\ x^*\mathscr{F}_j & \overset{\simeq}{\dashrightarrow} & y^*\mathscr{F}_j \end{array} \tag{4}$$

commute, where the vertical maps are the transition maps. To see this, simply note that the set of $z \in Z$ for which there is a commutative diagram

$$\begin{array}{ccc} x^*\mathscr{F}_i & \overset{\simeq}{\dashrightarrow} & z^*\mathscr{F}_i \\ \downarrow & & \downarrow \\ x^*\mathscr{F}_j & \overset{\simeq}{\dashrightarrow} & z^*\mathscr{F}_j \end{array}$$

is a clopen subset of $X$, since any point admits a neighborhood on which both $\mathscr{F}_i$ and $\mathscr{F}_j$ are constant. Since all of the $\mathscr{F}_i$ have compact stalks by Theorem 2.3, it follows from Lemma 3.3 that the stalk $(x_*C)_y \simeq \mathrm{colim}_I\, y^*\mathscr{F}_i$ is nonzero. But $X$ is Hausdorff, so this implies that $y = x$ as desired. □

**Remark 3.4.** Lemma 3.3 is also true if $\mathcal{C}$ is any ordinary category, e.g. the category of abelian groups Ab. It is illuminating to consider why the lemma holds in this concrete setting. Given a filtered system of abelian groups $\{A_i\}_{i \in I}$, the associated colimit can be described as the quotient of $\bigoplus_I A_i$ by the subgroup consisting of elements $a - \varphi_{ij}(a)$ where $a \in A_i$ and $\varphi_{ij} \colon A_i \to A_j$ is the transition map for some $j \geq i$. Clearly $\mathrm{colim}_I\, A_i \cong 0$ is implied by the assumption that for every $i \in I$ there is $j \geq i$ with $\varphi_{ij} \colon A_i \to A_j$ being zero. For the partial converse, assume now that each $A_i$ is a compact object of Ab, i.e. a finitely generated abelian group, and that $\mathrm{colim}_I\, A_i \cong 0$. Let $i \in I$ and pick a generating set $a_1, \ldots, a_n$ for $A_i$. Since $\mathrm{colim}_I\, A_j \cong 0$, there is $j_1, \ldots, j_n \geq i$ with $\varphi_{ij_s}(a_s) = 0$ for each $s$. Using that $I$ is filtered, pick $j \in I$ so that $j \geq j_s$ for each $s$. Then $\varphi_{ij}(a_s) = \varphi_{j_s j}\varphi_{ij_s}(a_s) = 0$ for each $s$, and hence $\varphi_{ij} = 0$.

3.2. **Hausdorff schemes.** Unlike in point-set topology, compactly generated categories of sheaves are abundant in algebraic geometry. Recall that a topological space $X$ is said to be *locally spectral* if it is homeomorphic to the underlying space of a scheme, and simply *spectral* if it is homeomorphic to the underlying space of an affine scheme. Using results of Hochster [Hoc69], Proposition 3.1 can be interpreted as saying that $\mathrm{Shv}(X, \mathcal{C})$ is only compactly generated when $X$ happens to come from algebraic geometry:

**Proposition 3.5.** *Let $\mathcal{C}$ be a nontrivial compactly generated stable $\infty$-category, and let $X$ be a $\mathcal{C}$-hypercomplete locally compact (resp. compact) Hausdorff space Then* $\mathrm{Shv}(X, \mathcal{C})$ *is compactly generated if and only if $X$ is locally spectral (resp. spectral).*

Indeed, a locally compact Hausdorff space is totally disconnected if and only if it admits a basis of compact open sets if and only if it is the underlying space of a scheme. The second equivalence is the Hausdorff case of [Hoc69, Thm 9].

For the first equivalence, note in one direction that if $X$ admits a basis of compact open sets, then every $x \in X$ has $\{x\} = \bigcap_{U \ni x} U$, with $U$ ranging over compact open neighborhoods of $x$. Since each compact open neighborhood is clopen, we thus have that $\{x\}$ is a quasi-component in $X$, and hence that $X$ is totally disconnected.

For the other direction, we must show that for every $x \in X$ and every open neighborhood $V \ni x$, there is a compact open $W$ with $x \in W \subseteq V$. Since $X$ is locally compact, we may assume that $V$ is precompact. By assumption $\{x\} = \bigcap_{U \ni x} U$, with $U$ ranging over clopen neighborhoods of $x$. Since each of these $U$ is in particular closed, we have that each $U \cap \partial \overline{V}$ is compact. By the finite intersection property, it therefore follows from $\bigcap_{U \ni x} U \cap \partial \overline{V} = \varnothing$ that for small enough clopen $U \ni x$, $U \cap \partial \overline{V} = \varnothing$. Hence $U \cap \overline{V} = U \cap V$ is a compact open neighborhood of $x$ contained in $V$, as desired.

3.3. **When is** $\mathrm{Shv}(X)$ **compactly generated?** Proposition 3.1 says that the $\infty$-category of sheaves on $X$ with coefficients in a stable $\infty$-category is rarely compactly generated when $X$ is a locally compact Hausdorff space. If we had asked the same question 'without coefficients,' this would have been an easier observation:

**Proposition 3.6.** *Let $X$ be a quasi-separated*[4] *topological space. The $\infty$-topos* $\mathrm{Shv}(X)$ *of sheaves of anima on $X$ is compactly generated if and only if the sobrification of $X$ is the underlying space of a scheme.*

*Proof.* One direction is [Lur09, Thm 7.2.3.6]. For the other direction, assume that $\mathrm{Shv}(X)$ is compactly generated. Then so is the frame $\mathcal{U} \simeq \tau_{\leq -1} \mathrm{Shv}(X)$ of open subsets of $X$ by [Lur09, Cor 5.5.7.4]. But this means that $X$ admits a basis of compact open sets, and hence the sobrification of $X$ is the underlying space of a scheme according to [Hoc69, Thm 9]. □

## Appendix A. Descent for maps with local sections

In this short appendix, we prove a descent lemma that was used in the proof of Theorem 2.3, which is an immediate generalization of [SD72, Cor 4.1.6].

Let $\mathcal{C}$ be a compactly generated $\infty$-category and let $f\colon X \to Y$ be a continuous map of topological spaces. Recall that the *Čech nerve* of $f$ is the augmented simplicial topological space $X_\bullet$ with $X_{-1} = Y$ and $p$-simplices

$$X_p = \underbrace{X \times_Y \cdots \times_Y X}_{p \text{ times}}$$

for $p \geq 0$, with face maps given by projections and degeneracy maps given in the obvious way. More formally, if $\Delta_+$ is the category of finite (possibly empty) ordinals and $\mathcal{T}\mathrm{op}$ is the category of topological spaces, then $X_\bullet \colon \Delta_+^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$ is defined by right Kan extending $(f\colon X \to Y)\colon \Delta_{+,\leq 0}^{\mathrm{op}} \to \mathcal{T}\mathrm{op}$ along the inclusion functor $\Delta_{+,\leq 0}^{\mathrm{op}} \subset \Delta_+^{\mathrm{op}}$.

Letting $\mathrm{Shv}^*(-,\mathcal{C})$ denote the contravariant functor from $\mathcal{T}\mathrm{op}$ to $\widehat{\mathcal{C}\mathrm{at}}_\infty$ given informally by $X \mapsto \mathrm{Shv}(X,\mathcal{C})$ on objects and $f \mapsto f^*$ on morphisms, we then have the following useful definition:

**Definition A.1.** The function $f$ is of $\mathcal{C}$*-descent type* if the canonical functor

$$\mathrm{Shv}(X,\mathcal{C}) \to \lim_\Delta \mathrm{Shv}^*(X_\bullet,\mathcal{C})$$

is an equivalence.

Let us say that $f$ *admits local sections* if for every $x \in X$, there is an open set $U \ni x$ such that the restriction $f\colon f^{-1}(U) \to U$ admits a section.

[4]Recall that a topological space $X$ is said to be *quasi-separated* if for any pair of compact open subsets $U, V \subseteq X$, the intersection $U \cap V$ is again compact. Note that all Hausdorff spaces are quasi-separated.

**Proposition A.2.** *If $f$ admits local sections, then $f$ is of $\mathcal{C}$-descent type.*

*Proof.* By ordinary Čech descent, we may assume that $f$ admits a section globally on $X$, after possibly passing to an open cover of $X$ on which this is true. Let $\varepsilon\colon Y \to X$ be a choice of such a section. The section $\varepsilon$ allows us to endow the Čech nerve $X_\bullet$ with the structure of a split augmented simplicial space, by defining the extra degeneracies $h_i\colon X_p \to X_{p+1}$ by

$$h_i(x_0, \dots, x_p) = (x_0, \dots, x_{i-1}, \varepsilon(y), x_i, \dots, x_p)$$

where $y = f(x_0) = \dots = f(x_p)$. It then follows that the split coaugmented cosimplicial $\infty$-category $\mathrm{Shv}^*(X_\bullet, \mathcal{C})$ is a limit diagram by [Lur09, Lem 6.1.3.16] □

**Corollary A.3.** *Let $\{A_i\}_{i\in I}$ be a collection of subsets of $X$ such that $X = \bigcup_I A_i^\circ$, where $A_i^\circ$ is the interior of $A_i$. Then the canonical map $\coprod_I A_i \to X$ is of $\mathcal{C}$-descent type.*

*Proof.* The canonical map $\coprod_I A_i \to X$ admits a section on $A_j^\circ$ given by $A_j^\circ \hookrightarrow A_j \to \coprod_I A_i$, where the second map is the canonical injection. □

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