Reconstruction of topological spaces from sheaf categories

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Purdue Topology Seminar (online)
October 28, 2026



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Plan

1. Motivation and main result
2. Dualizable categories
3. Reconstruction of spaces

$\mathrm{Shv}(X;\mathrm{D}(k))$: (∞/dg)-category of derived sheaves of k-modules on topological space X

Question

Let X and Y be nice ($=$ triangularizable) topological spaces with

$$\mathrm{Shv}(X;\mathrm{D}(\mathbb{Z})) \simeq \mathrm{Shv}(Y;\mathrm{D}(\mathbb{Z})). \tag{1}$$

Does it follow that X and Y are homeomorphic?

- Say that X and Y are *(non-trivial) FM partners* if (they are homeomorphic and) there is an equivalence (1)
- Without a niceness assumption, there are many stupid examples of non-trivial FM partners
- For nice spaces, the answer to the question is unknown

Motivation: FM partners in algebraic geometry

For the analogous question for schemes and DQCoh, there is a rich theory both on the positive side

Theorem (Bondal–Orlov 2001)

Let X and Y be smooth projective varieties, and suppose that either ω_X (anti-Fano) or ω_X^{-1} (Fano) is ample. If $\mathrm{DQoh}(X) \simeq \mathrm{DQoh}(Y)$, then $X \cong Y$.

. . . and the negative side

Examples of non-trivial FM partners in algebraic geometry

- An abelian variety and its dual (Mukai 1981)
- Smooth projective threefolds related by a flop (Bridgeland 2002)
- etc

Main theorem

Theorem (H. 2026)

The following spaces do not admit non-trivial FM partners:

- *second-countable one dimensional spaces (i.e. graphs);*
- *(finite type surfaces[a]); and*
- *the 3-sphere.*

[a] Sergey Guminov found a mistake in the proof, which I am working on fixing

- **Idea**: Use that for locally compact Hausdorff X, the category $\mathrm{Shv}(X; \mathrm{D}(\mathbb{Z}))$ has an intrinsic duality

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Interlude: Presentable categories I

Definition (Lurie, Adámek–Rosický)

An ∞-category $\mathcal{C}$ is *presentable* if it satisfies one of the following equivalent definitions:

- $\mathcal{C} \simeq \mathrm{Ind}_\kappa(K)$, for some regular cardinal κ and small κ-cocomplete ∞-category K; and
- $\mathcal{C}$ is the localization of a combinatorial model category at its weak equivalences.

A presentable ∞-category is called *κ-compactly generated* if it satisfies the second condition for given κ

Examples

Almost all the large categories we care about are presentable ($\mathrm{D}(A)$, $\mathcal{S}$, Sp, $\mathrm{Shv}(X; \mathrm{D}(\mathbb{Z}))$, $\mathrm{DQCoh}(X)$ ...)

Interlude: Presentable categories II

All presentable categories live together harmoniously in the same very large category:

Construction (Lurie)

There is a (very large) closed symmetric monoidal ∞-category $(\mathcal{P}\mathrm{r}^{\mathrm{L}}, \otimes, \mathcal{S})$, whose

- objects are presentable ∞-categories;
- morphisms are colimit-preserving functors;
- monoidal unit is the ∞-category of spaces/∞-groupoids $\mathcal{S}$; and
- internal hom is given by $(\mathcal{C}, \mathcal{D}) \mapsto \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ (the ∞-category of colimit-preserving functors)

The Lurie tensor product $\otimes$ has really nice properties:

- $\mathrm{Shv}(X; \mathcal{C}) \otimes \mathcal{D} \simeq \mathrm{Shv}(X; \mathcal{C} \otimes \mathcal{D})$
- $\mathrm{Mod}_R \otimes \mathrm{Mod}_S \simeq \mathrm{Mod}_{R \otimes S}$ for E_∞-rings R and S
- $\mathrm{D}(R) \otimes_{\mathrm{D}(k)} \mathrm{D}(S) \simeq \mathrm{D}(R \otimes^{\mathbb{L}}_k S)$ for dg k-algebras R and S

Verdier duality I

$\mathcal{P}\mathrm{r}^{\mathrm{L}}_{k,\mathrm{st}} = \mathrm{Mod}_{\mathrm{D}(k)}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ = the closed symmetric monoidal ∞-category of k-linear stable presentable ∞-categories, with relative Lurie tensor product $\otimes = \otimes_k$.

Definition

A *dualizable k-linear category* is a dualizable object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{k,\mathrm{st}}$.

A dualizable k-linear category $\mathcal{C}$ has essentially unique data

$$\mathcal{C}^\vee \in \mathcal{P}\mathrm{r}^{\mathrm{L}}_{k,\mathrm{st}}, \quad \mathrm{coev}\colon \mathrm{D}(k) \longrightarrow \mathcal{C} \otimes \mathcal{C}^\vee, \quad \mathrm{ev}\colon \mathcal{C}^\vee \otimes \mathcal{C} \longrightarrow \mathrm{D}(k),$$

witnessing $\mathcal{C}^\vee$ as dual of $\mathcal{C}$.

Theorem (Lurie)

If $\mathcal{C}$ is a compactly-generated k-linear category (such as the derived category of a qcqs scheme or of a dg algebra), then $\mathcal{C}$ is dualizable.

Verdier duality II

$\mathrm{Shv}(X;\mathrm{D}(k))$ is almost never compactly-generated:

Theorem (Neeman, H., Efimov)

$\mathrm{Shv}(X;\mathrm{D}(k))$ is compactly-generated if and only if X is totally disconnected.
An object $F \in \mathrm{Shv}(X;\mathrm{D}(k))$ is compact if and only if

- *F has compact support $\{x \in X \mid F_x \not\simeq 0\}$;*
- *$F_x \in \mathrm{Perf}(k)$ for all $x \in X$; and*
- *F is locally constant.*

Verdier duality III

. . . nevertheless:

Theorem (Lurie, Verdier)

$\mathrm{Shv}(X;\mathrm{D}(k))$ *is self-dual, as witnessed by*

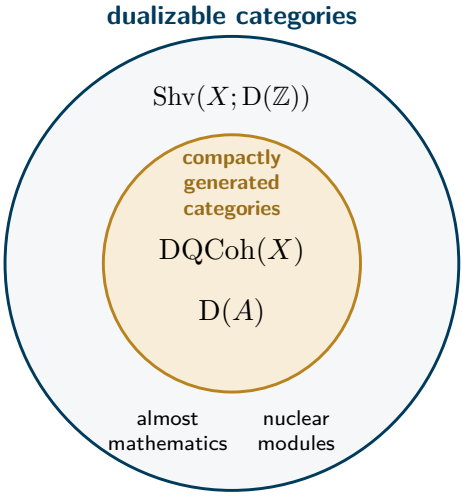
$$\mathrm{coev}\colon \mathrm{D}(k) \xrightarrow{\Delta_* p^*} \mathrm{Shv}(X\times X;\mathrm{D}(\mathbb{Z})) \simeq \mathrm{Shv}(X;\mathrm{D}(k))\otimes \mathrm{Shv}(X;\mathrm{D}(k))$$

and

$$\mathrm{ev}\colon \mathrm{Shv}(X;\mathrm{D}(k))\otimes \mathrm{Shv}(X;\mathrm{D}(k)) \simeq \mathrm{Shv}(X\times X;\mathrm{D}(\mathbb{Z})) \xrightarrow{p_!\Delta^*} \mathrm{D}(k).$$

Verdier duality IV

Overarching goal: extend tools from derived non-commutative geometry (Orlov, Bondal, Kontsevich, etc) from the compactly-generated to the dualizable setting



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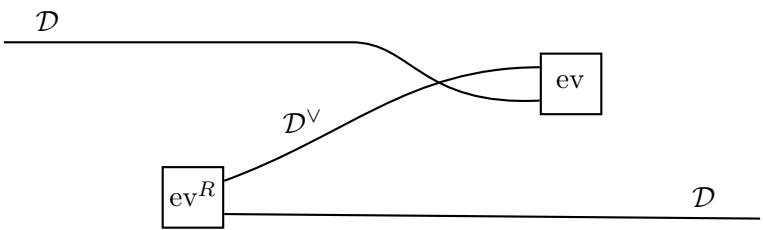
3 Reconstruction of spaces

The Serre functor

Construction (Bondal–Kapranov, Lurie)

Let $\mathcal{D}$ be a dualizable category. The *Serre functor* of $\mathcal{D}$ is the endofunctor

$$S_{\mathcal{D}} \colon \mathcal{D} \simeq \mathcal{D} \otimes \mathrm{D}(k) \xrightarrow{\mathrm{id}\otimes \mathrm{ev}^{\mathrm{R}}} \mathcal{D} \otimes \mathcal{D}^{\vee} \otimes \mathcal{D} \simeq \mathcal{D}^{\vee} \otimes \mathcal{D} \otimes \mathcal{D} \xrightarrow{\mathrm{ev}\otimes \mathrm{id}} \mathrm{D}(k) \otimes \mathcal{D} \simeq \mathcal{D}.$$



The Gorenstein locus

Construction (H.)

Let $\mathcal{D}$ be a dualizable category. The *Gorenstein locus* of $\mathcal{D}$, denoted $\mathcal{D}_{\mathrm{Gor}}$, is the full subcategory consisting of $x \in \mathcal{D}$ such

$$\mathrm{Hom}_{\mathcal{D}}(x, y) \xrightarrow{S_{\mathcal{D}}} \mathrm{Hom}_{\mathcal{D}}(Sx, Sy)$$

is an equivalence for all $y \in \mathcal{D}$.

Intuitively, the Gorenstein locus consists of objects that are "supported" only at "points" near which the Serre functor is invertible

The Gorenstein locus of sheaves

Definition (Goresky–MacPherson)

For a triangularizable space X, the *k-homology manifold locus* of X, denoted X_{hmfd}, is given by

$$X_{\mathrm{hmfd}} = \{x \in X \mid \omega_X|_U \simeq k_U[d] \text{ for some open } U \ni x\},$$

where ω_X is the Verdier dualizing sheaf.

Theorem (H.)

Let k be a field and let X be a triangularizable space. Then

$$\begin{aligned}\mathrm{Shv}(X;\mathrm{D}(k))_{\mathrm{Gor}} &= \{F \in \mathrm{Shv}(X;\mathrm{D}(k)) \mid F_x \not\simeq 0 \Leftrightarrow x \in X_{\mathrm{hmfd}}\} \\ &\simeq \mathrm{Shv}(X_{\mathrm{hmfd}};\mathrm{D}(k))\end{aligned}$$

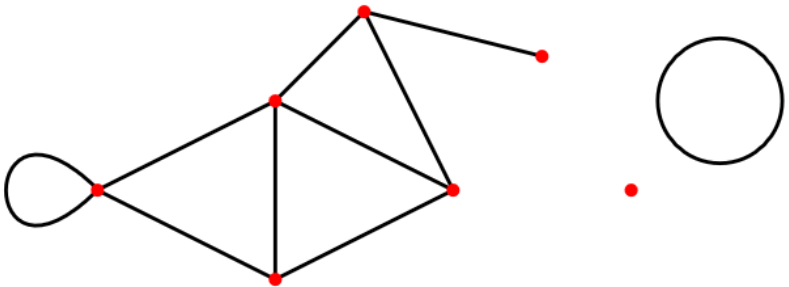
Reconstruction of graphs

Slogan

The reconstruction problem for nice spaces breaks into reconstruction for homology manifolds + categorical extension problems

Corollary (H.)

If X is second-countable and one-dimensional, then X has no non-trivial FM partners.



Tate–Hochschild cohomology I

Theorem (Lurie)

A presentable stable category $\mathcal{D}$ is dualizable if and only if the functor

$$\varinjlim\colon \operatorname{Ind}(\mathcal{D}) \longrightarrow \mathcal{D}$$

has a left adjoint $\hat{y}$.

- The functor $\varinjlim$ always has a *right* adjoint, namely the Yoneda embedding $y\colon \mathcal{D} \to \mathcal{D}$.
- For formal reasons y and $\hat{y}$ are fully faithful and there is a natural transformation

$$\theta_{\mathcal{D}}\colon \hat{y} \longrightarrow y.$$

Tate–Hochschild cohomology II

Definition (Eu–Schedler, Jørgensen–Bergh, . . . , H.)

For a dualizable stable category $\mathcal{D}$, the *Tate–Hochschild cohomology* of $\mathcal{D}$, denoted $\mathrm{HH}_{\mathrm{sg}}(\mathcal{D})$, is the image of

$$\operatorname{cofib}\left(\hat{y}_{\mathcal{D}\otimes\mathcal{D}^{\vee}}(\operatorname{coev}_{\mathcal{D}}(k)) \xrightarrow{\theta_{\mathcal{D}\otimes\mathcal{D}^{\vee}}} y_{\mathcal{D}\otimes\mathcal{D}^{\vee}}(\operatorname{coev}_{\mathcal{D}}(k))\right) \in \operatorname{Ind}(\mathcal{D}\otimes\mathcal{D}^{\vee})$$

under

$$\operatorname{Ind}(\mathcal{D}\otimes\mathcal{D}^{\vee}) \xrightarrow{\operatorname{Ind}(\operatorname{coev}^{\mathrm{R}}_{\mathcal{D}})} \operatorname{Ind}(\mathrm{D}(k)) \xrightarrow{\varinjlim} \mathrm{D}(k).$$

Theorem (H.)

Let A be a dg k-algebra. If the diagonal module $A \in \mathrm{D}(A \otimes^{\mathbb{L}}_{k} A^{\mathrm{op}})$ admits a (possibly infinite) resolution by perfect complexes of $A \otimes^{\mathbb{L}}_{k} A^{\mathrm{op}}$-modules, then the definition of $\mathrm{HH}_{\mathrm{sg}}(\mathrm{D}(A))$ given above recovers the established definition of Tate–Hochschild cohomology.

Tate–Hochschild cohomology III

Intuition: $\mathrm{HH}_{\mathrm{sg}}(\mathrm{Shv}(X;\mathrm{D}(k)))$ should have something to do with ends of X, at least for nice X.

Incorrect theorem (H.)

Let k be a commutative ring. For a second countable locally compact triangularizable d-dimensional k-homology manifold X, there is an equivalence

$$\mathrm{HH}_{\mathrm{sg}}(\mathrm{Shv}(X;\mathrm{D}(k))) \simeq \mathrm{fib}\left(\mathrm{R}\Gamma_{\mathrm{c}}(X,\omega_X) \longrightarrow \mathrm{R}\Gamma(X,\omega_X)\right)[-2d+1] \tag{2}$$

in $\mathrm{D}(k)$.

- For nice X, the right-hand side above is a shift of the cochains of the "end space" of X as considered by Quinn and others
- Unfortunately, the proof and maybe also the statement is incorrect (as discovered by Sergey Guminov), so I am working on correcting this